VON MANGOLDT'S FORMULA

For clarity, let me give you explicitly the version of Mangoldt's formula to be used in our proof of the Prime Number Theorem as well as in our discussion of the relevance of the zeros of the Riemann zeta function:

$$\psi(x) = x - \sum_{\rho=\beta+i\gamma:|\gamma|<T} \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log(1-x^{-2}) + R(x,T),$$

where the sum is over the nontrivial zeros ρ of $\zeta(s)$ and

$$R(x,T) \ll \frac{x(\log xT)^2}{T} + \log x.$$

Note that we have replaced $\psi^*(x)$ by $\psi(x)$. This is OK because the difference between these two functions is absorbed by the term $\log x$ in the bound for $R(x,T)$.

I do not expect you to be able prove this formula, but you should understand

- that the formula comes from the inverse of the Mellin transform of $\psi^*(x)$
- that we by extending the integration along a vertical path to a rectangular contour going "deeply" into the left half-plane, can use the residue theorem
- how each of the terms in the formula is related to the poles of $\zeta'(s)/(s\zeta(s))$, and in particular that the first term x comes from the pole of $\zeta(s)$ at $s=1$
- the significance of the nontrivial zeros of $\zeta(s)$ both in the proof of the formula and in the formula itself

Remark. It is possible to get rid of the term $\log x$ in the bound for $R(x,T)$ and obtain the exact formula

$$\psi^*(x) = x - \lim_{T\to\infty} \sum_{\rho=\beta+i\gamma:|\gamma|<T} \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log(1-x^{-2}),$$

but the formula for $\psi(x)$ given above is much more useful because it only involves zeros up to height T.