NOTES ON THE GAMMA FUNCTION AND THE RIEMANN
ZETA FUNCTION

1. Some results from analysis

Lemma 1. Suppose \((f_n)\) is a sequence of functions analytic on an open subset \(D\) of \(\mathbb{C}\). If \((f_n)\) converges uniformly on every compact (closed and bounded) subset of \(D\) to the limit function \(f\) then \(f\) is analytic on \(D\). Moreover, the sequence of derivatives \((f'_n)\) converges uniformly on compact subsets of \(D\) to \(f'\).

Proof. Since \(f_n\) is analytic on \(D\) we have by Cauchy’s integral formula

\[
f_n(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f_n(z)}{z-w} \, dz
\]

where \(\gamma\) is any closed and positively oriented contour in \(D\) and \(w\) is any interior point. The region interior to and including \(\gamma\) is closed and bounded and hence compact. So \((f_n)\) converges uniformly on this region and hence we can pass to the limit under the integral sign giving

\[
f(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-w} \, dz.
\]

This implies that \(f\) is analytic on the region defined by \(\gamma\) and hence on the whole of \(D\).

For the derivatives we have

\[
f'_n(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'_n(z)}{(z-w)^2} \, dz
\]

and

\[
f'(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-w)^2} \, dz.
\]

Hence

\[
|f'_n(w) - f'(w)| = \frac{1}{2\pi} \left| \int_{\gamma} \frac{f_n(z) - f(z)}{(z-w)^2} \, dz \right|
\]

\[
\leq \text{(length of } \gamma) \times \sup_{z \in \gamma} \left| \frac{f_n(z) - f(z)}{(z-w)^2} \right|
\]

and this tends to 0 as \(n \to \infty\) for any \(w\) on the interior of \(\gamma\), and hence for any compact subsets of \(D\) (just choose \(\gamma\) appropriately which is possible since \(D\) is open). \(\square\)
Lemma 2 (Differentiating under the integral sign). Let $D$ be an open set and let $\gamma$ be a contour of finite length $L(\gamma)$. Suppose $\varphi : \{\gamma\} \times D \rightarrow \mathbb{C}$ is a continuous function and define $g : D \rightarrow \mathbb{C}$ by

$$g(z) = \int_{\gamma} \varphi(w, z)dw.$$  

Then $g$ is continuous. Also, if $\frac{\partial \varphi}{\partial z}$ exists and is continuous on $\{\gamma\} \times D$ then $g$ is analytic with derivative

$$g'(z) = \int_{\gamma} \frac{\partial \varphi}{\partial z}(w, z)dw.$$  

Proof. Let $z_1, z_2 \in D$ with $z_2$ fixed. Since $\varphi$ is continuous, given $\varepsilon > 0$ we can find a $\delta > 0$ such that $|z_1 - z_2| < \delta \Rightarrow |\varphi(w, z_1) - \varphi(w, z_2)| < \varepsilon/L(\gamma)$. Hence, given $\varepsilon > 0$ choose $\delta$ as above then by linearity of the integral and the estimation lemma

$$|g(z_1) - g(z_2)| = \left| \int_{\gamma} (\varphi(w, z_1) - \varphi(w, z_2))dw \right| \leq L(\gamma) \max_{w \in \gamma} |\varphi(w, z_1) - \varphi(w, z_2)| < \varepsilon.$$  

Hence $g$ is continuous. If $\frac{\partial \varphi}{\partial z}$ exists and is continuous then

$$\lim_{h \to 0} \left| \frac{\varphi(w, z + h) - \varphi(w, z)}{h} - \frac{\partial \varphi}{\partial z}(w, z) \right| = 0$$  

with $h$. Then again by linearity of the integral and the estimation lemma

$$\left| \frac{g(z + h) - g(z)}{h} - \int_{\gamma} \frac{\partial \varphi}{\partial z}(w, z)dw \right| = \left| \int_{\gamma} \left( \frac{\varphi(w, z + h) - \varphi(w, z)}{h} - \frac{\partial \varphi}{\partial z}(w, z) \right)dw \right| \leq L(\gamma) \max_{w \in \gamma} \left| \frac{\varphi(w, z + h) - \varphi(w, z)}{h} - \frac{\partial \varphi}{\partial z}(w, z) \right|$$

and this tends to 0 with $h$. □

Corollary 3. Let $D$ be an open set and $\varphi : [a, \infty] \times D \rightarrow \mathbb{C}$ be continuous with continuous partial derivative $\frac{\partial \varphi}{\partial z}$. If the integral $\int_{a}^{\infty} \varphi(t, z)dt$ converges uniformly on compact subsets of $D$ then it defines an analytic function there and has derivative $\int_{a}^{\infty} \frac{\partial \varphi}{\partial z}(t, z)dt$.

Proof. Let $f_n(z) = \int_{a}^{n} \varphi(t, z)dt$ (so $\gamma$ is the straight line joining $a$ and $n$). By the above lemma each $f_n$ is analytic (with $f'_n(z) = \int_{a}^{n} \frac{\partial \varphi}{\partial z}(t, z)dt$) and by hypothesis
\( f_n \to f = \int_a^\infty \varphi(t, z)dt \) uniformly on compact subsets of \( D \). Applying Lemma 1 gives us the result. □

2. The Gamma function

Let \( s = \sigma + it \) with \( \sigma, t \in \mathbb{R} \). We define the \( \Gamma \)-function for \( \sigma > 0 \) by

\[
\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt.
\]

Note that in the integral we have the two 'bad' points: 0 and \( \infty \). Also, we cannot immediately apply Corollary 3 since the integrand is not always continuous at 0. It turns out none of this matters and we have the following.

**Proposition 4.** \( \Gamma(s) \) is analytic for \( \sigma > 0 \).

**Proof.** First, note for \( a > 0 \) the function defined by \( \int_a^\infty e^{-t} t^{s-1} dt \) is analytic. To see this we only need show it is uniformly convergent (on compact blah) and then we can apply Corollary 3 since all other hypotheses are met. As expected, the exponential dominates the tail of the integral giving

\[
\left| \int_a^n e^{-t} t^{s-1} dt - \int_a^\infty e^{-t} t^{s-1} dt \right| = \left| \int_n^\infty e^{-t} t^{s-1} dt \right| 
\leq \int_n^\infty e^{-t} dt 
\leq C \int_n^\infty e^{-\frac{1}{2} t} dt 
= 2Ce^{-\frac{1}{2}n}
\]

and this → 0 as \( n \to \infty \) giving uniform convergence. Now for \( \sigma > 0 \) define,

\[
f_n(s) = \int_{\frac{1}{n}}^\infty e^{-t} t^{s-1} dt.
\]

By the above argument each \( f_n \) is analytic. Suppose \( \sigma \geq c > 0 \). For \( 0 < t \leq 1 \) we have \( e^{-t} < 1 \) and \( t^{\sigma-1} \leq t^{c-1} \). Hence, for \( n > m \),

\[
\left| \int_{\frac{1}{n}}^{\frac{1}{m}} e^{-t} t^{s-1} dt \right| < \int_{\frac{1}{n}}^{\frac{1}{m}} t^{c-1} dt = \frac{1}{c} (m^{-a} - n^{-a}).
\]

Given \( \varepsilon > 0 \) we can choose \( 0 < \delta < 1 \) such that \( \frac{1}{c} (m^{-a} - n^{-a}) < \varepsilon \) whenever \( |m^{-1} - n^{-1}| < \delta \). Hence the \( f_n \) satisfy the Cauchy condition for uniform convergence in compact subsets of the halfplane \( \sigma > 0 \). Applying Lemma 1 we see that the gamma function is analytic for \( \sigma > 0 \). □
We can show the $\Gamma$ function is an extension of factorial function to complex arguments, via the following functional equation

**Proposition 5.** For $\sigma > 0$ we have

$$\Gamma(s + 1) = s\Gamma(s).$$

**Proof.** Integration by parts gives

$$\int_0^\infty e^{-t}s dt = -e^{-t}s\bigg|_0^\infty + \int_0^\infty e^{-t}s-1 dt = s\Gamma(s)$$

By direct computation we see $\Gamma(1) = 1$ and hence by induction $\Gamma(n+1) = n!$ for all positive integers $n$. The functional equation also gives us the analytic continuation of $\Gamma$.

**Theorem 1.** The $\Gamma$ function can be extended over the whole complex plane to a meromorphic function with simple poles at the negative integers and zero. The residues of these poles are given by

$$\text{Res}_{s=-n}(\Gamma(s)) = \frac{(-1)^n}{n!}$$

**Proof.** By (2) we have

$$\Gamma(s) = \frac{\Gamma(s+n)}{s(s+1)(s+2)\cdots(s+n-1)}$$

for any positive integer $n$. Now $\Gamma(s+n)$ is analytic for $\sigma > -n$ so the function on the right is meromorphic for $\sigma > -n$ with simple poles at $0, -1, -2, \ldots, -(n-1)$. Since $n$ is arbitrary we’re done. By construction this extension of $\Gamma$ satisfies (2). To calculate the residues we rewrite (4) as

$$\Gamma(s) = \frac{\Gamma(s+n+1)}{s(s+1)(s+2)\cdots(s+n)}$$

and proceed directly viz:

$$\text{Res}(\Gamma; -n) = \lim_{s \to -n} \frac{(s+n)\Gamma(s+n+1)}{s(s+1)(s+2)\cdots(s+n)}$$

$$= \lim_{s \to -n} \frac{\Gamma(s+n+1)}{s(s+1)(s+2)\cdots(s+n-1)}$$

$$= \frac{(-1)^n}{n!}$$

where we have used $\Gamma(1) = 1$ in the numerator.
From now on when we refer to the $\Gamma$-function we mean the meromorphic continuation. Heuristically we can think of this as the limit in $n$ of the right hand side of (4), and this is in fact not too far from the truth.

2.1. Product Representations of $\Gamma(s)$. Since $e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$ it is not unreasonable to expect

\begin{equation}
\Gamma(s) = \lim_{n \to \infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{s-1} dt.
\end{equation}

We first prove this and then use it to give an alternative representation of $\Gamma(s)$, which can be thought of as the limit in $n$ of (4).

**Lemma 6.** Formula (5) holds for $\sigma > 0$.

**Proof.** Denote

$$f_n(s) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{s-1} dt.$$

Then

$$\Gamma(s) - f_n(s) = \int_0^n \left(e^{-t} - \left(1 - \frac{t}{n}\right)^n\right) t^{s-1} dt + \int_n^\infty e^{-t} t^{s-1} dt.$$

The second integral is just the tail of the $\Gamma$-function which $\to 0$ as $n \to \infty$. We want to show the first integral also $\to 0$. This seems feasible since the first factor of the integrand gets arbitrarily small for increasing $n$. Now, for $0 \leq y \leq 1$ we have $1 + y \leq e^y \leq (1 - y)^{-1}$. For $n$ large set $y = t/n$ then

$$\left(1 - \frac{t}{n}\right)^n \leq e^{-t} \leq \left(1 + \frac{t}{n}\right)^{-n}.$$

Hence

$$0 \leq e^{-t} - \left(1 - \frac{t}{n}\right)^n = e^{-t} \left(1 - e^t \left(1 - \frac{t}{n}\right)^n\right) \leq e^{-t} \left(1 - \left(1 + \frac{t}{n}\right) \left(1 - \frac{t}{n}\right)^n\right) = e^{-t} \left(1 - \left(1 - \frac{t^2}{n^2}\right)^n\right).$$

Now, if $0 \leq a \leq 1$ then $(1 - a)^n \geq 1 - na$ when $na < 1$. Letting $a = t^2/n^2$ then for large $n$

$$1 - \left(1 - \frac{t^2}{n^2}\right)^n \leq \frac{t^2}{n}.$$
Therefore
\[ 0 \leq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leq n^{-1} t^2 e^{-t}. \]

This gives
\[ |\Gamma(s) - f_n(s)| \leq \frac{1}{n} \int_0^n e^{-t} t^{\sigma+1} dt < \frac{1}{n} \Gamma(\sigma + 2) \to 0 \]
as \( n \to \infty \) since \( \Gamma(\sigma + 2) \) is finite. \( \square \)

We deduce the alternative representation of \( \Gamma(s) \) as follows. Substituting \( u = t/n \) we have
\[
\int_0^n \left(1 - \frac{t}{n}\right)^n t^{s-1} dt = n^s \int_0^1 (1 - u)^n u^{s-1} du
\]
\[= n^s \left( \frac{1}{s} u^s (1 - u)^n \right|_0^1 + \frac{n}{s} \int_0^1 (1 - u)^{n-1} u^s du \]
\[= n^s \left( \frac{n}{s} \int_0^1 (1 - u)^{n-1} u^s du \right) \]
\[= \ldots \]
\[= n^s \frac{n(n-1) \ldots 1}{s(s+1) \ldots ((s-1) + n)} \int_0^1 u^{(s-1)+n} du \]
\[= \frac{n!}{s(s+1) \ldots (s+n)} n^s. \]

Taking the limit as \( n \to \infty \) gives

**Proposition 7.** For \( s \neq 0, -1, \ldots \)

\[ \Gamma(s) = \lim_{n \to \infty} \frac{n!}{s(s+1) \ldots (s+n)} n^s. \]

This converges for all other \( s \) so gives us another meromorphic continuation of \( \Gamma \).

This formula is quite useful and has a few consequences. The first of which is

**Corollary 8** (Weierstrass Product). For \( s \neq 0, -1, \ldots \) we have

\[ \Gamma(s) = \frac{e^{-\gamma s}}{s} \prod_{k=1}^\infty \left(1 + \frac{s}{k}\right)^{-1} e^{\gamma/k} \]

where \( \gamma \) is the Euler-Mascheroni constant.
Proof. For \( s \neq 0, -1, \ldots \) we have
\[
\Gamma(s) = \lim_{n \to \infty} \frac{n!}{s(s + 1) \cdots (s + n)} n^s
\]

\[
= \lim_{n \to \infty} \frac{1}{s(1 + s)(1 + s/2) \cdots (1 + s/n)} e^{s \log n}
\]

\[
= \lim_{n \to \infty} \frac{e^{s(\log n - 1 - \frac{1}{2} - \cdots - \frac{1}{n})}}{s(1 + s)(1 + s/2) \cdots (1 + s/n)} e^{s(1 + \frac{1}{2} + \cdots + \frac{1}{n})}
\]

\[
= \frac{e^{-\gamma s}}{s} \lim_{n \to \infty} \prod_{k=1}^{n} \left( 1 + \frac{s}{k} \right)^{-1} e^{|z/k|}.
\]

This formula clearly demonstrates the poles as well as giving us the fact that \( \Gamma(s) \) has no zeros. Also, taking the logarithm of the product, differentiating and then evaluating at \( s = 1 \) gives \( \Gamma'(1) = -\gamma \).

2.2. The Reflection and Duplication Formulae. We can use (6) to prove the famous reflection and duplication formulae. For the first of these we need a lemma.

**Lemma 9.** We have the following expansions

\[
\pi \cot(\pi s) = \frac{1}{s} + 2s \sum_{n=1}^{\infty} \frac{1}{s^2 - n^2}
\]

and

\[
\frac{\sin(\pi s)}{\pi s} = \prod_{n=1}^{\infty} \left( 1 - \frac{s^2}{n^2} \right).
\]

**Proof.** Let
\[F(s) = \frac{1}{s} + 2s \sum_{n=1}^{\infty} \frac{1}{s^2 - n^2}.\]

with \( s \in \mathbb{C} \setminus \mathbb{Z}. \) If \( s \) is near an integer we expect to see some fairly large terms in the series but these will die out as \( n \) increases. This is enough to guarantee absolute convergence: for \( n > \frac{1}{\sqrt{2}} |s| \) we have \( |s^2 - n^2| \geq n^2 - |s|^2 > n^2/2 \) and hence

\[
\sum_{n > \frac{1}{\sqrt{2}} |s|} \frac{1}{|s^2 - n^2|} < \sum_{n > \frac{1}{\sqrt{2}} |s|} \frac{1}{n^2}
\]

which converges. Adding in the finite number of other terms gives that the series in (8) converges absolutely. Note this also implies the series converges uniformly
on compact subsets and hence defines an analytic function on \( \mathbb{C} \setminus \mathbb{Z} \) by Lemma 1. Splitting the summand into partial fractions we see \( F(s) \) is periodic in \( \sigma \) with period 1. The pole at 0 is simple and has residue 1. By periodicity every poles is simple with residue 1. Therefore, the function defined by \( f(s) = \pi \cot(\pi s) - F(s) \) is entire and periodic in \( \sigma \) with period 1. We show \( f(s) \) is bounded then apply Liouville's theorem. By periodicity it suffices to show \( f \) is bounded when \( 0 \leq \sigma < 1 \) and since \( f \) is entire we need to show it's bounded as \( t = \Im(s) \to \pm\infty \).

\[
\pi \cot(\pi s) = \pi i \frac{e^{\pi is} + e^{-\pi is}}{e^{\pi is} - e^{-\pi is}} = \pi i + \frac{2\pi i}{e^{2\pi is} - 1}.
\]

Since \( |e^{2\pi is}| = e^{-2\pi t} \) we have \( \lim_{t \to \pm\infty} \pi \cot(\pi s) = \mp\pi i \). For \( F(s) \) note that in the region \( 0 \leq \sigma < 1 \) we have \( |t| \leq |s| < |t| + 1 \). We also have \( |s^2 - n^2| = |\sigma^2 - t^2 - n^2 + 2i\sigma t| \geq |\sigma^2 - t^2 - n^2| = |\sigma^2 - (t^2 + n^2)| \geq |t^2 + n^2| - \sigma^2 > t^2 + n^2 - 1 \).

Hence

\[
|F(s)| \leq \frac{1}{|t|} + 2(|t| + 1) \sum_{n=1}^{\infty} \frac{1}{n^2 + t^2 - 1} \\
\leq \frac{1}{|t|} + 2(|t| + 1) \int_{0}^{\infty} \frac{dx}{x^2 + t^2 - 1} \\
= \frac{1}{|t|} + 2(|t| + 1) \tan^{-1} \left( \frac{x}{\sqrt{t^2 - 1}} \right) \bigg|_{0}^{\infty} \\
= \frac{1}{|t|} + \pi \frac{|t| + 1}{\sqrt{t^2 - 1}}.
\]

So \( F(s) \) is bounded. Therefore \( f(s) \) is bounded and hence constant by Liouville. At \( s = 1/2 \) we have \( \pi \cot(\pi/2) = 0 \) and

\[
F(1/2) = 2 - \sum_{n=1}^{\infty} \left( \frac{1}{n - 1/2} - \frac{1}{n + 1/2} \right) = 0
\]

hence \( f \equiv 0 \).

To see (9) consider \( g(s) = \sin(\pi s)/(\pi s \prod_{n \geq 1}(1 - s^2/n^2)) \). The product is absolutely convergent so \( g \) exists for \( s \in \mathbb{C} \setminus \mathbb{Z} \). \( g(s) \) tends to 1 as \( s \) tends to 0 and \( g \) has period 1 implying \( g(s) \) tends to 1 as \( s \) tends to any integer. The logarithmic derivative is given by

\[
\frac{g'(s)}{g(s)} = \pi \cot(\pi s) - \left( \frac{1}{s} + 2s \sum_{n=1}^{\infty} \frac{1}{s^2 - n^2} \right) = 0
\]

hence \( g \) is constant and since \( g(0) = 1 \) we have \( g(s) = 1 \) for all \( s \). \( \square \)
Proposition 10 (Reflection Formula).

(10) \[ \Gamma(s)\Gamma(1 - s) = \frac{\pi}{\sin \pi s}. \]

Proof. By (6) and (9) we have

\[
\Gamma(s)\Gamma(-s) = \lim_{n \to \infty} \frac{n!}{s(s+1)\cdots(s+n)} \frac{n!}{-s(-s+1)\cdots(-s+n)} \frac{1}{n^s \cdot \cdots \cdot s^s \cdot \cdots \cdot (s+n)^s} = \frac{\pi}{\sin \pi s}.
\]

So by (2)

\[ \Gamma(s)\Gamma(1 - s) = \Gamma(s)(-s)\Gamma(-s) = \frac{\pi}{\sin \pi s}. \]

Setting \( s = 1/2 \) in the reflection formula gives \( \Gamma(1/2) = \sqrt{\pi} \).

Proposition 11 (Duplication Formula). We have the following formula

(11) \[ \Gamma(s)\Gamma \left( s + \frac{1}{2} \right) = 2^{1-2s}\pi^{1/2}\Gamma(2s) \]

Proof. The trick here is to use a convenient expression for \( \Gamma(2s) \). By (6) we have

\[
\frac{\Gamma(s)\Gamma(s+1/2)}{\Gamma(2s)} = \lim_{n \to \infty} \left\{ \frac{n!n^s}{s(s+1)\cdots(s+n)(s+1/2)(s+3/2)\cdots(s+1/2+n)} \right\} \times \frac{2s(2s+1)\cdots(2s+2n)}{(2n)!(2n)^{2s}}
\]

\[
= \frac{1}{2^{2s}} \lim_{n \to \infty} \left\{ \frac{(n!)^2 n^{1/2} 2^{2n+1}}{(2n)!(z+n+1/2)} \right\}
\]

\[
= \frac{1}{2^{2s}} \lim_{n \to \infty} \left\{ \frac{(n!)^2 2^{2n+1}}{(2n)!n^{1/2}(1+z/n+1/2n)} \right\}
\]

\[
= \frac{1}{2^{2s}} \lim_{n \to \infty} \left\{ \frac{(n!)^2 2^{2n+1}}{(2n)!n^{1/2}} \right\}
\]

\[
= \frac{1}{2^{2s}} C
\]

Setting \( s = 1/2 \) gives \( C = 2\Gamma(1/2) = 2\sqrt{\pi} \) and we’re done. \( \square \)
We finish this section on the Γ function with a formula that is very useful for estimating Γ(s).

2.3. **Stirling’s Formula.** The following theorem characterises the Γ function uniquely and will prove useful.

**Theorem 2 (Uniqueness Theorem).** Let F be analytic in the right half-plane \( \mathcal{A} = \{ s \in \mathbb{C} : \sigma > 0 \} \). Suppose \( F(s+1) = sF(s) \) and that \( F \) is bounded in the strip \( 1 \leq \sigma < 2 \). Then \( F(s) = a\Gamma(s) \) in \( \mathcal{A} \) with \( a = F(1) \).

**Proof.** Consider \( f(s) = F(s) - a\Gamma(s) \). The equation \( f(s+1) = sf(s) \) holds in \( \mathcal{A} \) and so we can extend \( f \) meromorphically to the whole plane as we did for Γ. If any poles occur these must be at the negative integers. Since \( f(1) = 0 \) we have \( \lim_{s \to 0} sf(s) = 0 \), hence \( f \) doesn’t have a pole, or anything worse, at 0 and we can thus continue \( f \) analytically to 0. This gives the analytic continuation of \( f \) to the negative integers via \( f(s+1) = sf(s) \).

Now, \( |\Gamma(s)| \leq \Gamma(\sigma) \) and this is bounded for \( 1 \leq \sigma < 2 \). Since \( F \) is bounded here by hypothesis, \( f \) is also. Now consider the region with \( 0 \leq \sigma \leq 1 \). If \( t = \Re(s) \leq 1 \) then \( f \) is bounded since it’s analytic here. If \( t > 1 \) then \( f \) is bounded here since \( f(s) = f(s+1)/s \) and \( f \) is bounded for \( 1 \leq \sigma < 2 \). Since \( f(s) \) and \( f(1-s) \) assume the same values for \( 0 \leq \sigma \leq 1 \) we have that \( g(s) = f(s)f(1-s) \) is bounded and analytic. By Liouville \( g(s) \equiv g(1) = 0 \) and hence \( f \equiv 0 \). □

Our goal is to use the uniqueness theorem to prove

\[
\Gamma(s) = \sqrt{2\pi} s^{s-\frac{1}{2}} e^{-s} e^{\mu(s)}
\]

for \( s \in \mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}_{\leq 0} \) where

\[
\mu(s) := - \int_0^\infty \frac{P_1(x)}{s+x} dx
\]

and \( P_1(x) = x - \lfloor x \rfloor - \frac{1}{2} \). We need to show \( \mu \) is analytic and that it possesses an appropriate functional equation (so that the above representation of Γ satisfies the functional equation (2)). For this we need a lemma.

**Lemma 12 (‘Twisted’ △ -inequality).** For \( s = re^{i\theta} \) and \( x \geq 0 \) we have

\[
|s+x| \geq (|s|+x) \cos(\theta/2).
\]

This gives

\[
|s+x| \geq (|s|+x) \sin(\delta/2)
\]

when \( |\theta| \leq \pi - \delta, \ 0 < \delta \leq \pi \) (since \( \cos(\theta/2) \geq \sin(\delta/2) \)).
Proof. Using \( \cos \theta = 1 - 2 \sin^2(\theta/2) \) and \( (r + x)^2 \geq 4rx \) we have
\[
|s + x|^2 = r^2 + 2rx \cos \theta + x^2 = (r + x)^2 - 4rx \sin^2(\theta/2) \geq (r + x)^2 \cos^2(\theta/2)
\]
\[\square\]

**Proposition 13.** \( \mu(s) \) is analytic in \( \mathbb{C}^- \).

Proof. Define \( Q(x) = \frac{1}{2}(x - \lfloor x \rfloor - (x - \lfloor x \rfloor)^2) \). Then \( Q(t) \) is an antiderivative of \( -P_1(x) \) (so continuous) and \( 0 \leq Q(x) \leq 1/8 \). We have
\[
-\int_{\alpha}^{\beta} \frac{P_1(x)}{s + x} \, dx = \frac{Q(x)}{s + x} \bigg|_{\alpha}^{\beta} + \int_{\alpha}^{\beta} \frac{Q(x)}{(s + x)^2} \, dx
\]
for \( 0 < \alpha < \beta < \infty \). Now let \( 0 < \delta \leq \pi \) and \( \varepsilon > 0 \). Then for \( x \geq 0, s = re^{i\theta} \) with \( r > \varepsilon \) and \( |\theta| \leq \pi - \delta \) we have (by the above lemma)
\[
\left| \frac{Q(x)}{(s + x)^2} \right| \leq \frac{|Q(t)|}{\sin^2(\delta/2)(\varepsilon + x)^2} \leq \frac{1}{8\sin^2(\delta/2)(\varepsilon + x)^2}.
\]
Hence the integral
\[
\int_0^{\infty} \frac{Q(x)}{(s + x)^2} \, dx
\]
is uniformly convergent in compact subsets of \( \mathbb{C}^- \) and therefore defines an analytic function by Corollary 3. But by (14) we have
\[
\mu(s) = \int_0^{\infty} \frac{Q(x)}{(s + x)^2} \, dx.
\]
\[\square\]

Note by (15) and (12), (13) we have
\[
|\mu(s)| \leq \frac{1}{8\cos^2(\theta/2)|s|}
\]
\[
|\mu(s)| \leq \frac{1}{8\sin^2(\delta/2)|s|}
\]
for \( s = re^{i\theta}, |\theta| \leq \pi - \delta, 0 < \delta \leq \pi \).

**Proposition 14.** For \( s \in \mathbb{C}^- \) we have
\[
\mu(s) - \mu(s + 1) = \left(s + \frac{1}{2}\right) \log \left(\frac{s + 1}{s}\right) - 1.
\]
Proof. Since \( P_1(x + 1) = P_1(x) \) we have
\[
\mu(s + 1) = - \int_0^\infty \frac{P_1(x)}{(s + 1) + x} \, dx = - \int_0^\infty \frac{P_1(x + 1)}{s + x + 1} \, dx
\]
\[
= - \int_1^\infty \frac{P_1(x)}{s + x} \, dx = \mu(s) - \left( - \int_0^1 \frac{x - |x| - 1/2}{s + x} \, dx \right)
\]
\[
= \mu(s) + \int_0^1 \frac{x - 1/2}{s + x} \, dx
\]
\[
= \mu(s) + \int_0^1 \left( 1 - \frac{s + 1/2}{s + x} \right) \, dx.
\]

\[\square\]

**Theorem 3** (Complex Stirling’s Formula). For \( s \in \mathbb{C}^- \) we have
\[
\Gamma(s) = \sqrt{2\pi} s^{s - \frac{1}{2}} e^{-s} e^{\mu(s)}.
\]
Proof. By Proposition 13 the function
\[
F(s) = s^{s - \frac{1}{2}} e^{-s} e^{\mu(s)}
\]
is analytic in \( \mathbb{C}^- \) (we define \( s^{s - \frac{1}{2}} = e^{(s - \frac{1}{2}) \log s} \) where \( \log \) is the principal branch of the logarithm). By Proposition 14 we have
\[
F(s + 1) = (s + 1)^{s + 1/2} e^{-s - 1} e^{\mu(s)} - (s + \frac{1}{2}) \log \left( \frac{s + 1}{s} \right) + 1 = s^{s + 1/2} e^{-s} e^{\mu(s)} = sF(s).
\]
Also \( F(s) \) is bounded in the region \( \mathcal{A} = \{ s \in \mathbb{C} : \sigma > 0 \} \). Clearly, \( e^{\mu(s)} \) is bounded by (16). Writing \( s = \sigma + it = |s| e^{i\theta} \in \mathbb{C} \) we have \( |s^{s - \frac{1}{2}} e^{-s}| = |s|^{\sigma - 1/2} e^{-\theta t} e^{-\sigma} \). Then for \( s \in \mathcal{A} \) with \( |t| \geq 2 \) we have \( \sigma - 1/2 \leq 2 \), \( |s| \leq 2t \) and \( -\theta t \leq -\pi |t|/2 \). Hence, in this region we have \( |s^{s - \frac{1}{2}} e^{-s}| \leq 4t^2 e^{-\pi|t|^2/2} e^{-1} \to 0 \) as \( |t| \to \infty \). Hence \( F(s) \) is bounded in \( \mathcal{A} \). By the Uniqueness Theorem we must have
\[
\Gamma(s) = a s^{s - \frac{1}{2}} e^{-s} e^{\mu(s)}
\]
for some \( a \). Substituting this into the duplication formula (11) gives
\[
a^2 s^{s - 1/2} e^{-s} e^{\mu(s)} (s + 1/2)^s e^{-s - 1/2} e^{\mu(s + 1/2)} = a 2^{1 - 2s} \sqrt{\pi} (2s)^{2s - 1/2} e^{-2s} e^{\mu(2s)}
\]
\[
= a \sqrt{2\pi} s^{2s - 1/2} e^{-2s} e^{\mu(2s)}
\]
Hence
\[
a(1 + 1/2s)^s e^{\mu(s) + \mu(s + 1/2)} = \sqrt{2\pi} e^{\mu(2s)}.
\]
Now let \( s \) be real and approach infinity. By (16) the exponentials tend to 1 and since \( \lim_{x \to \infty} (1 + 1/2x)^x = e \) we have \( a = \sqrt{2\pi} \). 

This result immediately gives the original form of stirling’s formula.
Corollary 15. As $x \to \infty$ in $\mathbb{R}$ we have $\Gamma(x) \sim \sqrt{2\pi}x^{x-1/2}e^{-x}$.

Another consequence is the following.

**Corollary 16** (Rapid decay in vertical strips). Let $\sigma \in \mathbb{R}$ be fixed. Then as $|t| \to \infty$ we have

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi}|t|^{|\sigma|-1/2}e^{-\pi|t|/2}.$$  

**Proof.** Assume $t \geq 0$. By (19) and (16) we have

$$\log |\Gamma(\sigma + it)| = \Re(\log(\Gamma(\sigma + it)))$$

$$= \Re((\sigma + it - 1/2)\log(\sigma + it) - \sigma - it) + \frac{1}{2}\log 2\pi + O(1/t)$$

$$= \Re((\sigma + it - 1/2)(\log(\sigma^2 + t^2)/2 + i\arg(\sigma + it)))$$

$$-\sigma + \frac{1}{2}\log 2\pi + O(1/t)$$

$$= (\sigma - 1/2)\left(\log(|t|\sqrt{1 + \sigma^2/t^2})\right) - t\arg(\sigma + it)$$

$$-\sigma + \frac{1}{2}\log 2\pi + O(1/t)$$

$$= (\sigma - 1/2)(\log |t| + o(1)) - t(\pi/2 - \tan^{-1}(\sigma/t))$$

$$-\sigma + \frac{1}{2}\log 2\pi + o(1)$$

$$= (\sigma - 1/2)\log |t| - \pi t/2 + \frac{1}{2}\log 2\pi + o(1).$$

Doing similar stuff for $t \leq 0$ gives the result. \qed

3. **The Riemann zeta function**

3.1. **Analytic continuation and functional equation.** Let $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$ and let

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}.$$  

The series is absolutely and locally uniformly convergent for $\sigma > 1$ and hence $\zeta(s)$ is analytic in this region. We plan to extend the domain on which $\zeta$ can be defined (hopefully this will allow us to exploit its connection with the primes). More precisely, we wish to find a function which agrees with the zeta function on the domain $\sigma > 1$,.
but that also makes sense outside of this region. This process is known as analytic continuation.

As an example of analytic continuation, consider the two functions

\[ f(z) = \sum_{n=0}^{\infty} z^n, \]

and

\[ g(z) = \frac{1}{1 - z}. \]

Now, \( f \) is analytic in the domain \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) and \( g \) is analytic in the larger domain \( z \in \mathbb{C} \setminus \{1\} \). However, the restriction of \( g \) to \( D \), \( g|_D \), is equal to \( f \) and so \( g \) is an analytic continuation of \( f \). Also, suppose we have another analytic continuation of \( f \) i.e. an analytic function \( h : \mathbb{C} \setminus \{1\} \to \mathbb{C} \) for which \( h|_D = f \). Then on \( D \) we must have \( g(z) - h(z) \equiv 0 \). Hence \( g(z) = h(z) \) for all \( z \in \mathbb{C} \setminus \{1\} \) since holomorphic functions are essentially determined by their local behaviour. Generalising these ideas we see that an analytic continuation of a given function is unique, and so we may talk of THE analytic continuation.

**Theorem 4** (Analytic continuation and functional equation). The Riemann zeta function can be continued to a function analytic on all of \( \mathbb{C} \) with the exception of a simple pole at \( s = 1 \) with residue 1. The continuation satisfies the functional equation

\[ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s). \]  

**Remark.** Here we have denoted the analytic continuation of the zeta function also by \( \zeta(s) \), really, this is the zeta function proper. When referring to the zeta function from now on we will mean its analytic continuation.

A key tool in our proof of Theorem 4 is the Poisson summation formula. In order to prove this, amongst other things, we need a suitable condition under which integrals and sums can be interchanged. This is given by a special case of Fubini’s Theorem.

**Proposition 17.** Suppose \( f_n \in L^1(\mathbb{R}) \) for all \( n \in \mathbb{Z} \) and that \( f_n(t) \in \ell^1 \) for all \( t \in \mathbb{R} \). If either

\[ \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} |f_n(t)| \, dt < \infty \quad \text{or} \quad \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |f_n(t)| \, dt < \infty, \]

then

\[ \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} |f_n(t)| \, dt = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |f_n(t)| \, dt. \]
Recall that the Schwartz space consists of smooth functions (infinitely differentiable) of rapid decay (derivatives of all order decay faster than any negative power of $x$: for all $k \geq 0$ and $m \geq 1$, $x^m f^{(k)}(x) \to 0$ as $|x| \to \infty$).

**Theorem 5** (Poisson summation). Suppose $f$ is in Schwartz space. Define the Fourier transform of $f$ by
\[
\tilde{f}(t) = \int_{\mathbb{R}} f(x)e^{-2\pi i tx}dx.
\]
Then
\[
\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \tilde{f}(n).
\]

**Proof.** We first periodise $f$. Let
\[
S(t) = \sum_{n \in \mathbb{Z}} f(n + t).
\]
Since $f$ is in Schwartz space, $S(t)$ is absolutely convergent. Also notice $S(t)$ is periodic of period $\leq 1$. Hence we can represent $S$ as a Fourier series
\[
S(t) = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i kt}.
\]
The Fourier coefficients $c_k$ are given by
\[
c_k = \int_{0}^{1} S(x)e^{-2\pi ikx}dx
\]  
  \[= \int_{0}^{1} \left( \sum_{n \in \mathbb{Z}} f(n + x) \right) e^{-2\pi ikx}dx \]
  \[= \sum_{n \in \mathbb{Z}} \int_{0}^{1} f(n + x)e^{-2\pi ikx}dx \]
  \[= \sum_{n \in \mathbb{Z}} \int_{n}^{n+1} f(t)e^{-2\pi ik(t-n)}dt \]
  \[= \int_{-\infty}^{\infty} f(t)e^{-2\pi ikt}dt \]
  \[= \tilde{f}(k). \]

Note the interchange of integration and summation is valid since $S(x)$ is absolutely convergent. Hence
\[
\sum_{n \in \mathbb{Z}} f(n + t) = \sum_{n \in \mathbb{Z}} \tilde{f}(n)e^{2\pi int}
\]
and setting $t = 0$ gives the result. \qed
Proof of Theorem 4. Our starting point is an identity involving the gamma function. Recall that
\[ \Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt. \]
Letting \( s \mapsto s/2 \) and substituting \( t = \pi n^2 x \) gives
\[ \int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx = n^{-s}\pi^{-s/2}\Gamma(s/2). \]
Hence for \( \sigma > 1 \),
\[ \pi^{-s/2}\zeta(s)\Gamma(s/2) = \sum_{n=1}^\infty \int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx = \int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^\infty e^{-\pi n^2 x}dx \]
if the interchange of summation and integration can be justified. This is clear since for \( \sigma > 1 \)
\[ \sum_{n=1}^\infty \int_0^\infty |x^{\frac{s}{2}-1}e^{-\pi n^2 x}|dx = \pi^{-\sigma/2}\zeta(\sigma)\Gamma(\sigma/2) < \infty. \]
We define
\[ \psi(x) = \sum_{n=1}^\infty e^{-\pi n^2 x} \]
so that
\[ \pi^{-s/2}\zeta(s)\Gamma(s/2) = \int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx \]
\[ = \int_0^1 x^{\frac{s}{2}-1}\psi(x)dx + \int_1^\infty x^{\frac{s}{2}-1}\psi(x)dx. \]
Since \( \psi(x) = O(e^{-\pi x}) \) we see by the usual argument that the second integral is absolutely convergent for all values of \( s \), and uniformly convergent on compact subsets of \( \mathbb{C} \). Since the integrand is differentiable the integral defines an analytic function by Corollary 3. We seek similar behaviour in the first integral so that we can give meaning to \( \zeta(s) \) for \( \sigma < 1 \). To this end we introduce
\[ \theta(x) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x} = 2\psi(x) + 1 \]
Writing $f(u) = e^{-\pi u^2}$ we see

$$
\tilde{f}(n) = \int_{-\infty}^{\infty} e^{-\pi u^2} e^{-2\pi i u n} du = \int_{-\infty}^{\infty} e^{-\pi x(u+in/x)^2} e^{-\pi n^2/x} du = e^{-\pi n^2/x} \int_{-\infty}^{\infty} e^{-\pi x u^2} du = \frac{1}{\sqrt{x}} e^{-\pi n^2/x}.
$$

Since $f$ is a Schwartz function we can apply Poisson summation to give

$$
\theta(x) = \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right)
$$

and hence

$$
2\psi(x) + 1 = \frac{1}{\sqrt{x}} (2\psi(1/x) + 1).
$$

Therefore for $\sigma > 1$ we obtain

$$
\pi^{-s/2} \zeta(s) \Gamma(s/2) = \int_1^1 \frac{1}{x^{s-1}} \left( \frac{1}{\sqrt{x}} \psi(1/x) + \frac{1}{2\sqrt{x}} - \frac{1}{2} \right) dx + \int_1^\infty x^{\frac{s}{2}-1} \psi(x) dx
$$

$$
= \frac{1}{s-1} - \frac{1}{s} + \int_0^1 x^{\frac{s}{2}-\frac{3}{2}} \psi(1/x) dx + \int_1^\infty x^{\frac{s}{2}-1} \psi(x) dx
$$

$$
= \frac{1}{s(s-1)} + \int_1^\infty \frac{1}{y^{s/2-1}} \psi(y) dy + \int_1^\infty x^{\frac{s}{2}-1} \psi(x) dx
$$

$$
= \frac{1}{s(s-1)} + \int_1^\infty \left( x^{\frac{s}{2}-1} + x^{\frac{1-s}{2}-1} \right) \psi(x) dx.
$$

Once again we see the integral represents an entire function and so the right hand side is analytic on $\mathbb{C}\{0, 1\}$. Since the factor $\pi^{s/2} \Gamma(s/2)$ never vanishes we may use the above expression to define $\zeta(s)$ for $\mathbb{C}\{0, 1\}$. Since $\Gamma(s/2)^{-1}$ has a simple zero at $s = 0$ we conclude the resulting expression for $\zeta(s)$ is analytic at $s = 0$. Also note that the expression on the right is invariant under $s \mapsto 1 - s$ and hence the functional equation follows. Finally, to find the residue at $s = 1$ we use (23) to calculate $\lim_{s \to 1} (s-1) \zeta(s)$ and note $\Gamma(1/2) = \sqrt{\pi}$. □

The nature of the functional equation

$$
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)
$$
allows us to infer the behaviour of $\zeta(s)$ for $\sigma < 0$ from its behaviour when $\sigma > 1$. In particular, the left hand side is non-zero and holomorphic for $\sigma > 1$. Since $\Gamma(s)$ has simple poles at the negative integers, $\zeta(s)$ must have simple zeros at the negative even integers, and these are the only zeros for $\sigma < 0$. We refer to these as the ‘trivial’ zeros of $\zeta(s)$. If $\zeta(s)$ were to have any other zeros then these must lie in the region $0 \leq \sigma \leq 1$, known as the critical strip.

3.2. **The Hadamard Product.** Similarly to the product expression for sin:

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right),$$

we wish to express the zeta function as a product over its zeros. Thus, we must first investigate infinite products in general.

### 3.2.1. Infinite products.

**Definition 1.** Let $(z_n)$ be a sequence of complex numbers and let

$$P(N) = \prod_{n=1}^{N} z_n.$$

Then if the limit $\lim_{N \to \infty} P(N)$ exists it is called the infinite product of $(z_n)$ and it is denoted $\prod_{n=1}^{\infty} z_n$.

If no term in the sequence $(z_n)$ is zero then $P(N)$ is non-zero for all $N$. If $\lim P(N) \neq 0$ then

$$\lim z_N = \lim \frac{P(N)}{P(N-1)} = 1.$$

Therefore, a necessary condition for the convergence of a non-zero infinite product is that the $n$th term goes to 1. Consequently, we may assume throughout the following that $\Re(z_n) > 0$ for all $n$. This allows us to take logarithms of the $z_n$ (using the principal branch) and hence convert products into sums.

**Proposition 18.** Suppose $\Re(z_n) > 0$ for all $n$. Then the infinite product $\prod_{n=1}^{\infty} z_n$ converges to a non-zero number if and only if $\sum_{n=1}^{\infty} \log z_n$ converges.

**Proof.** If $\sum_{n=1}^{N} \log z_n$ converges to some limit, $z$ say, then by continuity

$$\prod_{n=1}^{\infty} z_n = \lim_{N \to \infty} \exp \left( \sum_{n=1}^{N} \log z_n \right) = \exp \left( \lim_{N \to \infty} \sum_{n=1}^{N} \log z_n \right) = e^z \neq 0.$$
Suppose \( \prod_{n=1}^{\infty} z_n = z = re^{i\theta} \) with \(-\pi < \theta \leq \pi\) and \(r > 0\). Let

\[ P(N) = \prod_{n=1}^{N} z_n, \quad S(N) = \sum_{n=1}^{N} \log z_n \]

and let \( \log P(N) = \log |P(N)| + i\theta_N \) with \(-\pi + \theta < \theta_N \leq \pi + \theta\). Since \( P(N) = \exp(S(N)) \) we must have \( S(N) = \log P(N) + 2\pi i k_N \) for some integer \( k_N \). Now, since \( P(N) \to z \), we have \( z_N \to 1 \) and hence \( S(N) - S(N-1) = \log z_N \to 0 \). Also, \( \log P(N) - \log P(N-1) \to 0 \) which implies \( \theta_N - \theta_{N-1} \to 0 \) and so \( k_N - k_{N-1} \to 0 \).

Therefore, since \( k_N \) is an integer there exists some fixed integer \( k \) for which \( k_n = k \) for all \( n \) after some point \( n_0 \). Consequently, \( S(N) \to \log r + i\theta + 2\pi i k \).

\[ \square \]

**Definition 2.** If \( \Re(z_n) > 0 \) for all \( n \) we say the product \( \prod_{n=1}^{\infty} z_n \) converges absolutely iff the sum \( \sum_{n=1}^{\infty} \log z_n \) converges absolutely. We may always refer to finite products as absolutely convergent (no restriction on \( \Re(z_n) \) needed).

The following inequality will prove useful. By the power series expansion for the logarithm we have for \( |z| < 1 \)

\[
\left| 1 - \frac{\log(1+z)}{z} \right| = \left| \frac{z}{2} - \frac{z^2}{3} + \cdots \right| \\
\leq \frac{1}{2} (|z| + |z|^2 + \cdots ) \\
= \frac{1}{2} \frac{|z|}{1 - |z|}.
\]

If \( |z| < 1/2 \) then this last quantity is \( < 1/2 \). Hence, for \( |z| < 1/2 \) we have

\[ \frac{1}{2} |z| \leq |\log(1+z)| \leq \frac{3}{2} |z|. \]

This immediately gives the following.

**Proposition 19.** Suppose \( \Re z_n > -1 \). Then the series \( \sum_{n=1}^{\infty} \log(1 + z_n) \) converges absolutely if and only if \( \sum_{n=1}^{\infty} z_n \) converges absolutely.

**Examples.**

(i) The Euler product

\[ \prod_p \left( 1 - \frac{1}{p^s} \right)^{-1} \]

converges absolutely for \( \sigma > 1 \): This follows from \( \sum_p |p^{-s}| = \sum_p p^{-\sigma} < \sum_n n^{-\sigma} < \infty \).
(ii) The product $\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$ converges absolutely for all $z \in \mathbb{C}$: Fix $z \in \mathbb{C}$ and let $|z| = R$. Then the product $\prod_{n=1}^{2R} \left(1 - \frac{z^2}{n^2}\right)$ is finite and hence absolutely convergent. In the remaining product $\prod_{n=2R+1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$ we have $|z^2/n^2| < 1/2$ and hence $\Re(1 - z^2/n^2) > 0$. The result now follows from $\sum_{n>2R} z^2/n^2 \ll R^2$.

(iii) The product $\prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right)$ does not converge for any non-zero $z \in \mathbb{C}$: This follows from the divergence of the harmonic series.

(iv) The product $\prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n}$ converges absolutely for all $z \in \mathbb{C}$: Fix $z$ and choose $n_0$ such that $|z/n| < 1/2$ and hence $\Re((1 - z/n)e^{z/n}) > 0$ for all $n \geq n_0$. Then

$$\sum_{n=n_0}^{\infty} \log((1 - z/n)e^{z/n}) = \sum_{n=n_0}^{\infty} \log(1 - z/n) + z/n = - \sum_{n=n_0}^{\infty} \frac{z^2}{2n^2} + \frac{z^3}{3n^3} + \cdots \ll |z|^2 \sum_{n=n_0}^{\infty} \frac{1}{n^2} \left(1 + \frac{|z|}{n} + \frac{|z|^2}{n^2} + \cdots \right) \lessgtr |z|^2 \sum_{n=n_0}^{\infty} \frac{1}{n^2} \left(1 - \frac{|z|}{n}\right) \ll 2\zeta(2)|z|^2.$$

Similarly, for any sequence $(z_n)$ which satisfies $\sum_{n=1}^{\infty} |z_n|^{-2} < \infty$ the product $\prod_{n=1}^{\infty} (1 - z/n) e^{z_n/z_n}$ is absolutely convergent for all $z \in \mathbb{C}$. This is important because it allows us to create a function whose only zeros are at the points $z_n$. The main goal of this section is to show that this function is also analytic.

**Lemma 20.** Let $K$ be a compact (closed and bounded) subset of $\mathbb{C}$ and let $f, f_n : K \to \mathbb{C}$ be a set of functions such that $f_n(z) \to f(z)$ uniformly for $z \in K$. If there is a constant $A$ such that $\Re f(z) \leq A$ for all $z \in K$ then $\exp(f_n(z)) \to \exp(f(z))$ uniformly for $z \in K$.

**Proof.** Given $\epsilon > 0$ choose $\delta > 0$ such that $|e^z - 1| < \epsilon e^{-A}$ for $|z| < \delta$. Also, choose $N$ such that $|f_n(z) - f(z)| < \delta$ for all $z \in K$ whenever $n > N$. With these choices $\epsilon e^{-A} > |\exp(f_n(z)) - 1| = |\exp(f_n(z)) - \exp(f(z))| - 1|$. 
Therefore,

\[ |\exp(f_n(z)) - \exp(f(z))| < \epsilon e^{-A} |\exp(f(z))| < \epsilon. \]

\[ \square \]

Proposition 21. Let \((z_n)\) be a sequence of non-zero complex numbers and suppose \(\sum_{n=1}^{\infty} |z_n|^{-2}\) converges. Then the product

\[ P(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{z/z_n} \quad (25) \]

is absolutely and uniformly convergent on compact subsets of \(\mathbb{C}\) and is therefore analytic (by Lemma 1).

Proof. We have already shown the absolute convergence. It remains to show that

\[ S_m(z) = \sum_{n=1}^{m} \log((1 - z/z_n)e^{z/z_n}) \]

converges uniformly to

\[ S(z) = \sum_{n=1}^{\infty} \log((1 - z/z_n)e^{z/z_n}) \]

on compact subsets \(K\) and that also in such regions there exists an \(A\) such that \(\Re S(z) \leq A\). So fix a compact \(K\) and let \(\max_{z \in K} |z| = R\). Since we’ve already shown absolute convergence then it must follow that \(\Re S(z)\) is bounded for \(z \in K\). Also, following a similar analysis to before

\[ |S(z) - S_m(z)| = \left| \sum_{n=m+1}^{\infty} \log((1 - z/z_n)e^{z/z_n}) \right| \ll R^2 \sum_{n=m+1}^{\infty} \frac{1}{|z_n|^2} \to 0 \]

as \(m \to \infty\) and clearly this estimate holds uniformly for \(z \in K\). \[ \square \]

So given a sequence \((z_n)\) satisfying \(\sum |z_n|^{-2} < \infty\) we can construct an analytic function whose only zeros are the points \(z_n\). Conversely, given an analytic function \(f(z)\) with zeros \((z_n)\), we would like to know what conditions on \(f\) guarantee \(\sum |z_n|^{-2} < \infty\). Assuming that \(f\) satisfies these conditions we could then write

\[ f(z) = F(z)P(z) \]

where \(P(z)\) is given by (25) and where \(F(z)\) is manifestly some analytic function with no zeros. It turns out that the behaviour of a function’s zeros is closely related to its order of growth.
3.2.2. Entire functions of order 1.

**Definition 3.** A function is called *entire* if it is analytic on all of \( \mathbb{C} \). An entire function \( f(z) \) is said to be of *finite order* if there exists an \( \alpha \) such that

\[
f(z) = O(e^{\alpha|z|^\alpha}), \quad |z| \to \infty
\]

The order of \( f \) is defined as the infimum of all such \( \alpha \).

The behaviour of an analytic function zeroes and its order of growth are related through the following result of Jensen, which we quote without proof.

**Theorem 6 (Jensen’s formula).** Suppose \( f \) is analytic on the disc \( |z| \leq R \) and suppose \( z_1, \ldots, z_n \) are its zeros on the interior \( |z| < R \). Also, suppose \( f \) does not have any zeros on the boundary \( |z| = R \). Then

\[
\frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta - \log |f(0)| = \log \frac{R^n}{|z_1| \cdots |z_n|}
\]

**Corollary 22.** Let \( f \) be an entire function of order \( \alpha \) and let \( n(R) \) denote the number of zeros of \( f \) in \( |z| < R \). Then

\[
n(R) \ll R^\beta
\]

for any \( \beta > \alpha \).

**Proof.** Let \( z_1, \ldots, z_n \) denote the number of zeros of \( f \) in \( |z| < r \) and let \( |z_i| = r_i \). Then the right hand side of (26) can be written as

\[
\int_0^R \frac{n(r)}{r} dr
\]

since this is equal to

\[
\log r_2/r_1 + 2 \log r_3/r_2 + \cdots + n \log R/r_n = \log (R^n/r_1 \cdots r_n).
\]

Suppose \( \beta > \alpha \). Then

\[
\log |f(Re^{i\theta})| \ll R^\beta
\]

and hence

\[
\int_0^R \frac{n(r)}{r} dr \ll R^\beta - \log |f(0)| \ll R^\beta.
\]

On the other hand

\[
\int_R^{2R} \frac{n(r)}{r} dr \geq n(R) \int_R^{2R} \frac{1}{r} dr = n(R) \log 2.
\]

\( \square \)

**Corollary 23.** Suppose \( f \) is an entire function of order 1 with non-zero zeros \( (z_n) \). Then the series \( \sum_{n=1}^{\infty} |z_n|^{-1-\epsilon} \) converges for any \( \epsilon > 0 \).
Proof. By partial summation we have
\[
\sum_{n=1}^\infty |z_n|^{-1-\epsilon} = (1 + \epsilon) \int_0^\infty r^{-2-\epsilon} n(r)dr \ll \int_0^\infty r^{-1-\epsilon/2}dr < \infty.
\]

A consequence of this is that for an entire function \(f\) of order 1 with zeros \((z_n)\), the product
\[
P(z) = \prod_{n=1}^\infty \left(1 - \frac{z}{z_n}\right) e^{z/z_n}
\]
is an entire function. Writing
\[
f(z) = F(z)P(z)
\]
we see that \(F(z)\) is an entire function without zeros.

**Proposition 24.** Let \(f\) be an entire function of order 1 with zeros \((z_n)\) and let \(f(z) = F(z)P(z)\) where \(P(z)\) is given by (28). Then \(F(z)\) is a non-zero entire function of order at most 1.

Proof. We need to show that \(F\) has order at most 1. We do this by lower bounding \(P(z)\) on a sequence of circles \(|z| = R\). These must be kept away from the values \(r_n = |z_n|\), since otherwise we may obtain trivial lower bounds e.g. from an occurrence of \(1 - |z|/|z_n| = 0\). However, since \(\sum_{n=1}^\infty r_n^{-2}\) converges, these points cannot occur too densely. Indeed, the combined length of the intervals \((r_n - r_n^{-2}, r_n + r_n^{-2})\) is finite and hence we may always choose an arbitrarily large \(R\) such that
\[
|R - r_n| > r_n^{-2}
\]
for all \(n\).

Write \(P(z) = P_1(z)P_2(z)P_3(z)\) where the subproducts are extended over the following sets of \(n\):
\[
P_1 : \quad r_n < R/2
\]
\[
P_2 : \quad R/2 \leq r_n \leq 2R
\]
\[
P_3 : \quad r_n > 2R.
\]

Throughout the following, \(\epsilon\)'s may vary from line to line. For \(P_1\), on \(|z| = R\) we have
\[
|(1 - z/z_n)e^{z/z_n}| \geq (|z|/|z_n| - 1)e^{-|z|/|z_n|} > e^{-R/r_n}
\]
and since
\[
\sum_{r \leq R/2} r_n^{-1} = \sum_{r \leq R/2} r_n^\epsilon r_n^{-1-\epsilon} \leq (R/2)^\epsilon \sum_{n=1}^\infty r_n^{-1-\epsilon} \ll R^\epsilon
\]
it follows that
\[ P_1(z) \gg \exp(-R^{1+\epsilon}). \]

For \( P_2 \) we have
\[ |(1 - z/z_n)e^{z/z_n}| \geq |z/z_n - 1|e^{-|z|/|z_n|} \geq e^{-2|z - z_n|/2R} \gg R^{-3} \]
by (29). By (27), the number of factors in \( P_2 \) is \( \ll R^{1+\epsilon} \) and hence
\[ |P_2(z)| \gg (R^{-3})^{R^{1+\epsilon}} \gg \exp(-R^{1+2\epsilon}). \]

Finally, in \( P_3 \) we have \( |z/z_n| < 1/2 \) and hence
\[ |(1 - z/z_n)e^{z/z_n}| = |1 - \frac{z^2}{z_n^2} + O((z/z_n)^3)| > e^{-c(R/r_n)^2} \]
for some constant \( c > 0 \). Also,
\[ \sum_{n>2R} r_n^{-2} = \sum_{n>2R} r_n^{-1+\epsilon}r_n^{-1-\epsilon} < (2R)^{-1+\epsilon}\sum_{n=1}^{\infty} r_n^{-1-\epsilon} \ll R^{-1+\epsilon}. \]

Therefore,
\[ P_3(z) \gg \exp(-cR^{1+\epsilon}) \gg \exp(-R^{1+2\epsilon}). \]

Consequently, for \( |z| = R \) we have
\[ P(z) \gg \exp(-R^{1+\epsilon}) \]

and hence
\[ F(z) \ll \exp(R^{1+\epsilon}) \]
since \( f \) is of order 1. \( \square \)

**Proposition 25.** Let \( F(z) \) be an entire function of order \( \alpha \) with no zeros. Then \( F(z) = e^{G(z)} \) for some polynomial \( G \). The order of \( F \) is the degree of \( G \) and hence is an integer.

**Proof.** Since \( F \) is entire and non-zero the function \( G(z) := \log F(z) \) is itself entire and can be defined to be single valued. It satisfies
\[ \Re(G(z)) = \log |F(z)| \ll R^\alpha \]
for \( |z| = R \). Writing
\[ G(z) = \sum_{n=0}^{\infty} (a_n + ib_n)z^n \]
then
\[ \Re(G(z)) = \sum_{n=0}^{\infty} a_n R^n \cos n\theta - \sum_{n=1}^{\infty} b_n R^n \sin n\theta \]
for \( z = Re^{i\theta} \). Multiplying the above by \( 1 \pm \cos m\theta \) and integrating over \( \theta \in [0, 2\pi] \) gives \( \pi(2a_0 \pm a_m r^m) \) on the right side and \( O(R^\alpha) \) on the left side. Consequently,
Combining the above two Propositions with our previous results gives the following.

Theorem 7. Let \( f(z) \) be an entire function of order 1 with zeros \((z_n)\). Then

\[
f(z) = e^{a + bz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{z/z_n}
\]

for some constants \(a, b\). Moreover, the series \( \sum_{n=1}^{\infty} |z_n|^{-1-\epsilon} \) converges for all \( \epsilon > 0 \).

Note that if \( \sum_{n=1}^{\infty} |z_n|^{-1} \) converges then \( f(z) \leq e^{c|z|} \) for some constant \(c\). This follows on applying the inequality

\[
|(1 - z)e^{z}| = |1 - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{8} - \cdots| \leq e^{2|z|}
\]

to the product representation (31). Consequently, if the inequality \( f(z) \leq e^{c|z|} \) does not hold, then the series \( \sum_{n=1}^{\infty} |z_n|^{-1} \) diverges and so \( f \) must have infinitely many zeros.

3.3. The Hadamard product for the Riemann zeta function. Let

\[
\xi(s) = \frac{1}{2}s(s - 1)\pi^{-s/2}\Gamma(s/2)\zeta(s).
\]

Then this is an entire function whose only zeros are the non-trivial zeros of the zeta function. We will show that \( \xi(s) \) is of order 1. Since \( \xi(s) = \xi(1-s) \) we only need to look at the case \( \sigma \geq 1/2 \). Clearly,

\[
\frac{1}{2}s(s - 1)\pi^{-s/2} \ll \exp(c|s|)
\]

and by Stirling’s formula we have

\[
\Gamma(s/2) \ll \exp(c|s|\log|s|).
\]

On applying partial summation to the series representation of \( \zeta(s) \) when \( \sigma > 1 \) we find

\[
\zeta(s) = \frac{s}{s - 1} - s \int_{1}^{\infty} (x - \lfloor x \rfloor)x^{-s-1}dx.
\]

Note that the integral converges for \( \sigma > 0 \) and hence this representation acts as an analytic continuation of \( \zeta(s) \) to \( \sigma > 0 \). In particular, the integral is bounded for \( \sigma \geq 1/2 \) and therefore

\[
\zeta(s) \ll |s|
\]
for $|s|$ large. Putting these together gives

\[(33) \quad \xi(s) \ll \exp(c|s| \log |s|)\]

as $|s| \to \infty$. Note that since $\log \Gamma(s) \sim s \log s$ and $\zeta(s) \to 1$ the above inequality for $\xi(s)$ is essentially best possible. Therefore $\xi(s) = O(e^{c|s|^\alpha})$ for all $\alpha > 1$ and the infimum of all such $\alpha$ is 1 i.e. $\xi(s)$ is of order 1. In particular, $\xi(s)$ is not $\ll e^{c|z|}$ for any $c$. On applying the results of the previous section we have the following.

**Theorem 8.** The zeta function has an infinity of non-trivial zeros, $\rho_1, \rho_2, \ldots$, such that the series

\[\sum_{n=1}^{\infty} |\rho_n|^{-1-\epsilon}\]

converges for any $\epsilon > 0$ and

\[\sum_{n=1}^{\infty} |\rho_n|^{-1}\]

diverges. It satisfies the product representation

\[(34) \quad \zeta(s) = \frac{e^{A+Bs}}{(s-1)\Gamma(\frac{s}{2} + 1)} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}\]

for some constants $A, B$. 