Problems March 13, 2014

(1) $\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt = 2\int_0^\infty e^{-r^2} dr$

$= 2\left(\int_0^\infty \int_0^\infty e^{-x^2-y^2} dx\,dy\right)^{1/2}$

$= 2\left(\int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} r\, dr\, d\theta\right)^{\frac{1}{2}}$

$= \sqrt{\pi}\left(\int_0^\infty e^{-t} dt\right)^{\frac{1}{2}} = \underline{\underline{\sqrt{\pi}}}.$

(2) We find that

$\int_m^{m+1} (x - [x]) x^{-2} dx$

$= \log(m+1) - \log m - \frac{m}{m} + \frac{m}{m+1}$

$= \log(m+1) - \log m - \frac{1}{m+1}$. Thus

$\int_1^N (x-[x]) x^{-2} dx = 1 - \left(\sum_{m=1}^N \frac{1}{m} - \log N\right),$

and so $\int_1^\infty (x-[x])x^{-2} dx = 1 - \gamma$,

(3) $\zeta(s) = \frac{s}{s-1} - s\int_1^\infty (x-[x]) x^{-s-1} dx.$

It follows that $\zeta'(s) = -\frac{1}{(s-1)^2} + O(1)$

and $\zeta(s) = \frac{1}{s-1} + 1 - (1-\gamma) + O(s-1)$

when $s \to 1$. Hence:

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \gamma + G(s-1)$$

when $s \to 1$, and the result follows.

(4) The product formula for $\Gamma(s)$ gives

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\frac{1}{s} + \gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k+s} - \frac{1}{k}\right),$$

and so $\displaystyle\lim_{s\to 0} \frac{\Gamma'(s)}{\Gamma(s)} + \frac{1}{s} = -\gamma$.

Now the result follows from (3).

(5) It is clear from the product formula for $\Gamma(s)$ that

$$\Gamma(s)\Gamma(-s) = -\frac{1}{s^2}\prod_{k=1}^{\infty}\left(1-\frac{s^2}{k^2}\right)^{-1} = -\frac{\pi}{s\cdot\sin\pi s}.$$

Since $\Gamma(1-s) = -s\,\Gamma(-s)$, the result follows.

(6) The product formula gives

$$\frac{\Gamma(s)}{\Gamma(2s)} = 2\cdot e^{\gamma s}\prod_{j=0}^{\infty}\left(1+\frac{2s}{2j+1}\right)e^{-\frac{2s}{2j+1}}.$$

The function to the right has the same zeros as $1/\Gamma(s+\frac{1}{2})$. It follows that

$$\frac{\Gamma(\frac{1}{2})}{\Gamma(s+\frac{1}{2})} = e^{\gamma s + As} \prod_{j=0}^{\infty} \left(1 + \frac{2s}{2j+1}\right) e^{-\frac{2s}{2j+1}}.$$

Logarithmic differentiation gives

$$(s+\tfrac{1}{2})^{-1} + \sum_{k=1}^{\infty} \left(\frac{1}{s+k+\frac{1}{2}} - \frac{1}{k} \right)$$

$$= A + \sum_{j=0}^{\infty} \left(\frac{1}{s+j+\frac{1}{2}} - \frac{1}{j+\frac{1}{2}} \right).$$

This means that

$$A = 2\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \dots\right) = 2 \log 2,$$

and the result follows.

(7) We start from the functional equation

$$\pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}) \zeta(s) = \pi^{-\frac{1}{2}(1-s)} \Gamma(\tfrac{1-s}{2}) \zeta(1-s).$$

We first use (6) to obtain

$$\Gamma(\tfrac{s}{2}) = \pi^{\frac{1}{2}} 2^{1-s} \Gamma(s) / \Gamma(\tfrac{s+1}{2}).$$

By (5),

$$\Gamma(\tfrac{s+1}{2}) \Gamma(\tfrac{1-s}{2}) = \frac{\pi}{\sin \pi (\frac{s+1}{2})} = \frac{\pi}{\cos \frac{\pi}{2} s}.$$ Thus:

$$\zeta(1-s) = 2^{1-s} \pi^{-s} (\cos \tfrac{\pi}{2} s) \Gamma(s) \zeta(s).$$

(8) $$-\frac{\zeta'(1-s)}{\zeta(1-s)} = -\log 2 - \log \pi - \frac{\frac{\pi}{2}\sin\frac{\pi}{2}s}{\cos\frac{\pi}{2}s}$$

$$+\frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta'(s)}{\zeta(s)}.$$

Now it follows from (4) that

$$\frac{\zeta'(0)}{\zeta(0)} = \log(2\pi).$$

(9) We set $f(s) = x^s \zeta'(s)/(s\,\zeta(s))$.
We know that f has poles at $1, 0$, and all even negative integers.

$$\mathrm{Res}(f;1) = -x \qquad \text{by } (3)$$

$$\mathrm{Res}(f;0) = +\log(2\pi). \qquad (\text{by } (8)).$$

$$\mathrm{Res}(f;2k) = \frac{x^{-2k}}{2k} \qquad (\text{by } (8)),$$

(10) Since

$$\sum_{k=1}^{\infty} \frac{x^{-2k}}{2k} = \frac{1}{2}\log(1-x^{-2}), \quad \text{we get}$$

$$\frac{1}{2\pi i}\int_C \frac{x^s \zeta'(s)}{s\,\zeta(s)} = -x + \log(2\pi) + \frac{1}{2}\log(1-x^{-2})$$

$$+\sum_{0<|\mathrm{Im}\,\rho|<T} \frac{x^\rho}{\rho}.$$

(11) We have the formula from (4), and therefore it suffices to note that

$$\left| \sum_{k=1}^{\infty} \frac{1}{k+s} - \frac{1}{k} \right| \leq \sum_{k \leq 2|s|} \left(\frac{1}{|k+s|} + \frac{1}{k} \right)$$

$$+ \sum_{k \geq 2|s|} \frac{|s|}{|k+s| \cdot k} \leq 3 \log |s| + O(1).$$

(12) All terms in the formula from (8) are bounded when $\sigma \leq -1$, except the last two. It is therefore a direct consequence of (11) that

$$\left| \frac{\zeta'(s)}{\zeta(s)} \right| = O(\log |s| + 2)$$

when $\sigma \leq -1$.

We know from before that

$$\psi^*(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(- \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} \right) ds \, ;$$

what remains to establish is that

$$\psi^*(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log(2\pi) - \frac{1}{2} \log(1 - x^{-2})$$

(von Mangoldt's formula) is an estimate <u>across</u> the critical strip.