THE GAMMA FUNCTION

1. Some results from analysis

Lemma 1. *Suppose (f_n) is a sequence of functions analytic on an open subset D of $\mathbb{C}$. If (f_n) converges uniformly on every compact (closed and bounded) subset of D to the limit function f then f is analytic on D. Moreover, the sequence of derivatives (f_n') converges uniformly on compact subsets of D to f'.*

Proof. Since f_n is analytic on D we have by Cauchy's integral formula

$$f_n(w) = \frac{1}{2\pi i}\int_\gamma \frac{f_n(z)}{z-w}dz$$

where γ is any closed and positively oriented contour in D and w is any interior point. The region interior to and including γ is closed and bounded and hence compact. So (f_n) converges uniformly on this region and hence we can pass to the limit under the integral sign giving

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz.$$

This implies that f is analytic on the region defined by γ and hence on the whole of D.

For the derivatives we have

$$f_n'(w) = \frac{1}{2\pi i}\int_\gamma \frac{f_n(z)}{(z-w)^2}dz$$

and

$$f'(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^2}dz.$$

Hence

$$\begin{aligned}|f_n'(w) - f'(w)| &= \frac{1}{2\pi}\left|\int_\gamma \frac{f_n(z)-f(z)}{(z-w)^2}dz\right| \\ &\leq (\text{length of } \gamma) \times \sup_{z\in\gamma}\left|\frac{f_n(z)-f(z)}{(z-w)^2}\right|\end{aligned}$$

and this tends to 0 as $n \to \infty$ for any w on the interior of γ, and hence for any compact subsets of D (just choose γ appropriately which is possible since D is open). □

Lemma 2 (Differentiating under the integral sign)**.** *Let D be an open set and let γ be a contour of finite length $L(\gamma)$. Suppose $\varphi : \{\gamma\} \times D \to \mathbb{C}$ is a continuous function and define $g : D \to \mathbb{C}$ by*

$$g(z) = \int_\gamma \varphi(w, z)dw.$$

Then g is continuous. Also, if $\frac{\partial\varphi}{\partial z}$ exists and is continous on $\{\gamma\} \times D$ then g is analytic with derivative

$$g'(z) = \int_\gamma \frac{\partial\varphi}{\partial z}(w, z)dw.$$

Proof. Let $z_1, z_2 \in D$ with z_2 fixed. Since φ is continuous, given $\varepsilon > 0$ we can find a $\delta > 0$ such that $|z_1 - z_2| < \delta \Rightarrow |\varphi(w, z_1) - \varphi(w, z_2)| < \varepsilon/L(\gamma)$. Hence, given $\varepsilon > 0$ choose δ as above then by linearity of the integral and the estimation lemma

$$\begin{aligned} |g(z_1) - g(z_2)| &= \left| \int_\gamma (\varphi(w, z_1) - \varphi(w, z_2))dw \right| \\ &\leq L(\gamma) \max_{w\in\gamma} |\varphi(w, z_1) - \varphi(w, z_2)| \\ &< \varepsilon. \end{aligned}$$

Hence g is continuous. If $\frac{\partial\varphi}{\partial z}$ exists and is continuous then

$$\left| \frac{\varphi(w, z + h) - \varphi(w, z)}{h} - \frac{\partial\varphi}{\partial z}(w, z) \right| \to 0$$

with h. Then again by linearity of the integral and the estimation lemma

$$\begin{aligned} &\left| \frac{g(z + h) - g(z)}{h} - \int_\gamma \frac{\partial\varphi}{\partial z}(w, z)dw \right| \\ = &\left| \int_\gamma \left(\frac{\varphi(w, z + h) - \varphi(w, z)}{h} - \frac{\partial\varphi}{\partial z}(w, z) \right) dw \right| \\ \leq &\ L(\gamma) \max_{w\in\gamma} \left| \frac{\varphi(w, z + h) - \varphi(w, z)}{h} - \frac{\partial\varphi}{\partial z}(w, z) \right| \end{aligned}$$

and this tends to 0 with h. □

Corollary 1. *Let D be an open set and $\varphi : [a, \infty] \times D \to \mathbb{C}$ be continuous with continuous partial derivative $\frac{\partial\varphi}{\partial z}$. If the integral $\int_a^\infty \varphi(t, z)dt$ converges uniformly on compact subsets of D then it defines an analytic function there and has derivative $\int_a^\infty \frac{\partial\varphi}{\partial z}(t, z)dt$.*

Proof. Let $f_n(z) = \int_a^n \varphi(t, z)dt$ (so γ is the straight line joining a and n) . By the above lemma each f_n is analytic (with $f_n'(z) = \int_a^n \frac{\partial\varphi}{\partial z}(t, z)dt$) and by hypothesis $f_n \to f = \int_a^\infty \varphi(t, z)dt$ uniformly on compact subsets of D. Applying Lemma 1 gives us the result. □

2. The Analytic Character of $\Gamma(s)$

Let $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$. We define the Γ-function for $\sigma > 0$ by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt. \tag{1}$$

Note in the definition we have the two 'bad' points, 0 and ∞. Also, we cannot immediately apply Corollary 1 since the integrand is not always continuous at 0. It turns out none of this matters and we have the following.

Proposition 1. *$\Gamma(s)$ is analytic for $\sigma > 0$.*

Proof. First, note for $a > 0$ the function defined by $\int_a^\infty e^{-t} t^{s-1} dt$ is analytic. To see this we only need show it is uniformly convergent (on compact blah) and then we can apply Corollary 1 since all other hypotheses are met. As expected, the exponential dominates the tail of the integral giving

$$\begin{aligned} \left| \int_a^n e^{-t} t^{s-1} dt - \int_a^\infty e^{-t} t^{s-1} dt \right| &= \left| \int_n^\infty e^{-t} t^{s-1} dt \right| \\ &\leq \int_n^\infty e^{-t} t^{\sigma-1} dt \\ &\leq C \int_n^\infty e^{-\frac{1}{2}t} dt \\ &= 2Ce^{-\frac{1}{2}n} \end{aligned}$$

and this $\to 0$ as $n \to \infty$ giving uniform convergence. Now for $\sigma > 0$ define,

$$f_n(s) = \int_{\frac{1}{n}}^\infty e^{-t} t^{s-1} dt.$$

By the above argument each f_n is analytic. Suppose $\sigma \geq c > 0$. For $0 < t \leq 1$ we have $e^{-t} < 1$ and $t^{\sigma-1} \leq t^{c-1}$. Hence, for $n > m$,

$$\left| \int_{\frac{1}{n}}^{\frac{1}{m}} e^{-t} t^{s-1} dt \right| < \int_{\frac{1}{n}}^{\frac{1}{m}} t^{c-1} d = \frac{1}{c}(m^{-a} - n^{-a}).$$

Given $\varepsilon > 0$ we can choose $0 < \delta < 1$ such that $\frac{1}{c}(m^{-a} - n^{-a}) < \varepsilon$ whenever $|m^{-1} - n^{-1}| < \delta$. Hence the f_n satisfy the Cauchy condition for uniform convergence in compact subsets of the halfplane $\sigma > 0$. Applying Lemma 1 we see that the gamma function is analytic for $\sigma > 0$. $\square$

We can show the Γ function is an extension of factorial function to complex arguments, via the following functional equation

Proposition 2. *For $\sigma > 0$ we have*

$$\Gamma(s+1) = s\Gamma(s). \tag{2}$$

Proof. Integration by parts gives

$$\int_0^\infty e^{-t}t^s dt = -e^{-t}t^s\Big|_0^\infty + s\int_0^\infty e^{-t}t^{s-1}dt = s\Gamma(s)$$

□

By direct computation we see $\Gamma(1) = 1$ and hence by induction $\Gamma(n+1) = n!$ for all positive integers n. The functional equation also gives us the analytic continuation of Γ.

Theorem 1. *The Γ function can be extended over the whole complex plane to a meromorphic function with simple poles at the negative integers and zero. The residues of these poles are given by*

$$\text{Res}_{s=-n}\big(\Gamma(s)\big) = \frac{(-1)^n}{n!} \tag{3}$$

Proof. By (2) we have

$$\Gamma(s) = \frac{\Gamma(s+n)}{s(s+1)(s+2)\cdots(s+n-1)} \tag{4}$$

for any positive integer n. Now $\Gamma(s+n)$ is analytic for $\sigma > -n$ so the function on the right is meromorphic for $\sigma > -n$ with simple poles at $0, -1, -2, \ldots, -(n-1)$. Since n is arbitrary we're done. By construction this extension of Γ satisfies (2). To calculate the residues we rewrite (4) as

$$\Gamma(s) = \frac{\Gamma(s+n+1)}{s(s+1)(s+2)\cdots(s+n)}$$

and proceed directly viz:

$$\begin{aligned}\text{Res}(\Gamma;\, -n) &= \lim_{s\to -n} \frac{(s+n)\Gamma(s+n+1)}{s(s+1)(s+2)\cdots(s+n)} \\ &= \lim_{s\to -n} \frac{\Gamma(s+n+1)}{s(s+1)(s+2)\cdots(s+n-1)} \\ &= \frac{(-1)^n}{n!}\end{aligned}$$

where we have used $\Gamma(1) = 1$ in the numerator. □

From now on when we refer to the Γ-function we mean the meromorphic continuation. Heuristically we can think of this as the limit in n of the right hand side of (4), and this is in fact not too far from the truth.

3. Product Representations of $\Gamma(s)$

Since $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ it is not unreasonable to expect

$$\Gamma(s) = \lim_{n\to\infty}\int_0^n \left(1-\frac{t}{n}\right)^n t^{s-1}dt. \tag{5}$$

We first prove this and then use it to give an alternative representation of $\Gamma(s)$, which can be thought of as the limit in n of (4).

Lemma 3. *Formula (5) holds for $\sigma > 0$.*

Proof. Denote

$$f_n(s) = \int_0^n \left(1-\frac{t}{n}\right)^n t^{s-1}dt.$$

Then

$$\Gamma(s) - f_n(s) = \int_0^n \left(e^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^{s-1}dt + \int_n^\infty e^{-t}t^{s-1}dt.$$

The second integral is just the tail of the Γ-function which $\to 0$ as $n\to\infty$. We want to show the first integral also $\to 0$. This seems feasible since the first factor of the integrand gets arbitrarily small for increasing n. Now, for $0 \le y \le 1$ we have $1+y \le e^y \le (1-y)^{-1}$. For n large set $y = t/n$ then

$$\left(1-\frac{t}{n}\right)^n \le e^{-t} \le \left(1+\frac{t}{n}\right)^{-n}.$$

Hence

$$\begin{aligned} 0 \le e^{-t} - \left(1-\frac{t}{n}\right)^n &= e^{-t}\left(1 - e^t\left(1-\frac{t}{n}\right)^n\right) \\ &\le e^{-t}\left(1-\left(1+\frac{t}{n}\right)^n\left(1-\frac{t}{n}\right)^n\right) \\ &= e^{-t}\left(1-\left(1-\frac{t^2}{n^2}\right)^n\right). \end{aligned}$$

Now, if $0 \le a \le 1$ then $(1-a)^n \ge 1 - na$ when $na < 1$. Letting $a = t^2/n^2$ then for large n

$$1 - \left(1-\frac{t^2}{n^2}\right)^n \le \frac{t^2}{n}.$$

Therefore

$$0 \le e^{-t} - \left(1-\frac{t}{n}\right)^n \le n^{-1}t^2e^{-t}.$$

This gives

$$|\Gamma(s) - f_n(s)| \le \frac{1}{n}\int_0^n e^{-t}t^{\sigma+1}dt < \frac{1}{n}\Gamma(\sigma+2) \to 0$$

as $n \to \infty$ since $\Gamma(\sigma + 2)$ is finite. □

We deduce the alternative representation of $\Gamma(s)$ as follows. Substituting $u = t/n$ we have

$$\begin{aligned}
\int_0^n \left(1 - \frac{t}{n}\right)^n t^{s-1} dt &= n^s \int_0^1 (1-u)^n u^{s-1} du \\
&= n^s \left(\frac{1}{s} u^s (1-u)^n \Big|_0^1 + \frac{n}{s} \int_0^1 (1-u)^{n-1} u^s du \right) \\
&= n^s \left(\frac{n}{s} \int_0^1 (1-u)^{n-1} u^s du \right) \\
&= \cdots\cdots \\
&= n^s \frac{n(n-1)\ldots 1}{s(s+1)\ldots((s-1)+n)} \int_0^1 u^{(s-1)+n} du \\
&= \frac{n!}{s(s+1)\cdots(s+n)} n^s.
\end{aligned}$$

Taking the limit as $n \to \infty$ gives

Proposition 3. *For $s \neq 0, -1, \ldots$*

$$\Gamma(s) = \lim_{n\to\infty} \frac{n!}{s(s+1)\cdots(s+n)} n^s. \tag{6}$$

This converges for all other s so gives us another meromorphic continuation of Γ.

This formula is quite useful and has a few consequences. The first of which is

Corollary 2 (Weierstrass Product)**.** *For $s \neq 0, -1, \ldots$ we have*

$$\Gamma(s) = \frac{e^{-\gamma s}}{s} \prod_{k=1}^{\infty} \left(1 + \frac{s}{k}\right)^{-1} e^{z/k} \tag{7}$$

where γ is the Euler-Mascheroni constant.

Proof. For $s \neq 0, -1, \ldots$ we have

$$\begin{aligned}
\Gamma(s) &= \lim_{n\to\infty} \frac{n!}{s(s+1)\cdots(s+n)} n^s \\
&= \lim_{n\to\infty} \frac{1}{s(1+s)(1+s/2)\cdots(1+s/n)} e^{s\log n} \\
&= \lim_{n\to\infty} \frac{e^{s(\log n - 1 - \frac{1}{2} - \cdots - \frac{1}{n})}}{s} \frac{e^{s(1+\frac{1}{2}+\cdots+\frac{1}{n})}}{(1+s)(1+s/2)\cdots(1+s/n)} \\
&= \frac{e^{-\gamma s}}{s} \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{s}{k}\right)^{-1} e^{z/k}.
\end{aligned}$$

□

This formula clearly demonstrates the poles as well as giving us the fact that $\Gamma(s)$ has no zeros. Also, taking the logarithm of the product, differentiating and then evaluating at $s = 1$ gives $\Gamma'(1) = -\gamma$.

4. The Reflection and Duplication Formulae

We can use (6) to prove the famous reflection and duplication formulae. For the first of these we need a lemma.

Lemma 4. *We have the following expansions*

$$\pi\cot(\pi s) = \frac{1}{s} + 2s\sum_{n=1}^{\infty}\frac{1}{s^2 - n^2} \tag{8}$$

and

$$\frac{\sin(\pi s)}{\pi s} = \prod_{n=1}^{\infty}\left(1 - \frac{s^2}{n^2}\right). \tag{9}$$

Proof. Let

$$F(s) = \frac{1}{s} + 2s\sum_{n=1}^{\infty}\frac{1}{s^2 - n^2}.$$

with $s \in \mathbb{C}\backslash\mathbb{Z}$. If s is near an integer we expect to see some fairly large terms in the series but these will die out as n increases. This is enough to guarantee absolute convergence: for $n > \frac{1}{\sqrt{2}}|s|$ we have $|s^2 - n^2| \geq n^2 - |s|^2 > n^2/2$ and hence

$$\sum_{n > \frac{1}{\sqrt{2}}|s|}\frac{1}{|s^2 - n^2|} < \sum_{n > \frac{1}{\sqrt{2}}|s|}\frac{1}{n^2}$$

which converges. Adding in the finite number of other terms gives that the series in (8) converges absolutely. Note this also implies the series converges uniformly on compact subsets and hence defines an analytic function on $\mathbb{C}\backslash\mathbb{Z}$ by Lemma 1. Splitting the summand into partial fractions we see F(s) is periodic in σ with period 1. The pole at 0 is simple and has residue 1. By periodicity every poles is simple with residue 1. Therefore, the function defined by $f(s) = \pi\cot(\pi s) - F(s)$ is entire and periodic in σ with period 1. We show $f(s)$ is bounded then apply Liouville's theorem. By periodicity it suffices to show f is bounded when $0 \leq \sigma < 1$ and since f is entire we need to show it's bounded as $t = \Im(s) \to \pm\infty$. Now,

$$\pi\cot(\pi s) = \pi i\frac{e^{\pi i s} + e^{-\pi i s}}{e^{\pi i s} - e^{-\pi i s}} = \pi i + \frac{2\pi i}{e^{2\pi i s} - 1}.$$

Since $|e^{2\pi i s}| = e^{-2\pi t}$ we have $\lim_{t\to\pm\infty}\pi\cot(\pi s) = \mp\pi i$. For $F(s)$ note that in the region $0 \leq \sigma < 1$ we have $|t| \leq |s| < |t| + 1$. We also have $|s^2 - n^2| =$

$|\sigma^2 - t^2 - n^2 + 2i\sigma t| \geq |\sigma^2 - t^2 - n^2| = |\sigma^2 - (t^2 + n^2)| \geq |t^2 + n^2| - \sigma^2 > t^2 + n^2 - 1.$ Hence

$$\begin{aligned} |F(s)| &\leq \frac{1}{|t|} + 2(|t|+1)\sum_{n=1}^{\infty}\frac{1}{n^2+t^2-1} \\ &\leq \frac{1}{|t|} + 2(|t|+1)\int_0^{\infty}\frac{dx}{x^2+t^2-1} \\ &= \frac{1}{|t|} + 2(|t|+1)\frac{\tan^{-1}\left(x/\sqrt{t^2-1}\right)}{\sqrt{t^2-1}}\Bigg|_0^{\infty} \\ &= \frac{1}{|t|} + \pi\frac{|t|+1}{\sqrt{t^2-1}}. \end{aligned}$$

So $F(s)$ is bounded. Therefore $f(s)$ is bounded and hence constant by Liouville. At $s = 1/2$ we have $\pi\cot(\pi/2) = 0$ and

$$F(1/2) = 2 - \sum_{n=1}^{\infty}\left(\frac{1}{n-1/2} - \frac{1}{n+1/2}\right) = 0$$

hence $f \equiv 0$.

To see (9) consider $g(s) = \sin(\pi s)/(\pi s\prod_{n\geq 1}(1 - s^2/n^2))$. The product is absolutely convergent so g exists for $s \in \mathbb{C}\backslash\mathbb{Z}$. $g(s)$ tends to 1 as s tends to 0 and g has period 1 implying $g(s)$ tends to 1 as s tends to any integer. The logarithmic derivative is given by

$$\frac{g'(s)}{g(s)} = \pi\cot(\pi s) - \left(\frac{1}{s} + 2s\sum_{n=1}^{\infty}\frac{1}{s^2-n^2}\right) = 0$$

hence g is constant and since $g(0) = 1$ we have $g(s) = 1$ for all s. □

Proposition 4 (Reflection Formula)**.**

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}. \tag{10}$$

Proof. By (6) and (9) we have

$$\begin{aligned} \Gamma(s)\Gamma(-s) &= \lim_{n\to\infty}\frac{n!}{s(s+1)\cdots(s+n)}n^s\frac{n!}{-s(-s+1)\cdots(-s+n)}n^{-s} \\ &= \lim_{n\to\infty}\frac{1}{-z^2\prod_{k=1}^{n}\left(1+\frac{s}{n}\right)\left(1-\frac{s}{n}\right)} \\ &= -\frac{\pi}{s\sin\pi s}. \end{aligned}$$

So by (2)

$$\Gamma(s)\Gamma(1-s) = \Gamma(s)(-s)\Gamma(-s) = \frac{\pi}{\sin\pi s}.$$

□

Setting $s = 1/2$ in the reflection formula gives $\Gamma(1/2) = \sqrt{\pi}$.

Proposition 5 (Duplication Formula). *We have the following formula*

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = 2^{1-2s}\pi^{1/2}\Gamma(2s) \tag{11}$$

Proof. The trick here is to use a convenient expression for $\Gamma(2s)$. By (6) we have

$$\begin{aligned}
\frac{\Gamma(s)\Gamma(s+1/2)}{\Gamma(2s)} &= \lim_{n\to\infty}\left\{\frac{n!n^s}{s(s+1)\cdots(s+n)}\frac{n!n^{s+1/2}}{(s+\frac{1}{2})(s+\frac{3}{2})\cdots(s+\frac{1}{2}+n)}\right. \\
&\quad \left.\times\frac{2s(2s+1)\cdots(2s+2n)}{(2n)!(2n)^{2s}}\right\} \\
&= \frac{1}{2^{2s}}\lim_{n\to\infty}\left\{\frac{(n!)^2n^{1/2}2^{2n+1}}{(2n)!(z+n+\frac{1}{2})}\right\} \\
&= \frac{1}{2^{2s}}\lim_{n\to\infty}\left\{\frac{(n!)^2 2^{2n+1}}{(2n)!n^{1/2}(1+z/n+\frac{1}{2n})}\right\} \\
&= \frac{1}{2^{2s}}\lim_{n\to\infty}\left\{\frac{(n!)^2 2^{2n+1}}{(2n)!n^{1/2}}\right\} \\
&= \frac{1}{2^{2s}}C
\end{aligned}$$

Setting $s = 1/2$ gives $C = 2\Gamma(1/2) = 2\sqrt{\pi}$ and we're done. □

We finish this section on the Γ function with a formula that is very useful for estimating $\Gamma(s)$.

5. Stirling's Formula

The following theorem characterises the Γ function uniquely and will prove useful.

Theorem 2 (Uniqueness Theorem). *Let F be analytic in the right half-plane $\mathcal{A} = \{s \in \mathbb{C} : \sigma > 0\}$. Suppose $F(s+1) = sF(s)$ and that F is bounded in the strip $1 \le \sigma < 2$. Then $F(s) = a\Gamma$ in $\mathcal{A}$ with $a = F(1)$.*

Proof. Consider $f(s) = F(s) - a\Gamma(s)$. The equation $f(s+1) = sf(s)$ holds in $\mathcal{A}$ and so we can extend f meromorphically to the whole plane as we did for Γ. If any poles occur these must be at the negative integers. Since $f(1) = 0$ we have $lim_{s\to 0} sf(s) = 0$, hence f doesn't have a pole, or anything worse, at 0 and we can thus continue f analytically to 0. This gives the analytic continuation of f to the negative integers via $f(s+1) = sf(s)$.

Now, $|\Gamma(s)| \le \Gamma(\sigma)$ and this is bounded for $1 \le \sigma < 2$. Since F is bounded here by hypothesis, f is also. Now consider the region with $0 \le \sigma \le 1$. If $t = \Im(s) \le 1$ then f is bounded since it's analytic here. If $t > 1$ then f is bounded here since $f(s) = f(s+1)/s$ and f is bounded for $1 \le \sigma < 2$. Since $f(s)$ and $f(1-s)$ assume the same values for $0 \le \sigma \le 1$ we have that $g(s) = f(s)f(1-s)$ is bounded and analytic. By Liouville $g(s) \equiv g(1) = 0$ and hence $f \equiv 0$. $\square$

Our goal is to use the uniqueness theorem to prove

$$\Gamma(s) = \sqrt{2\pi} s^{s-\frac{1}{2}} e^{-s} e^{\mu(s)}$$

for $s \in \mathbb{C}^- = \mathbb{C} \backslash \mathbb{R}_{\le 0}$ where

$$\mu(s) := -\int_0^\infty \frac{P_1(x)}{s+x} dx$$

and $P_1(x) = x - \lfloor x \rfloor - \frac{1}{2}$. We need to show μ is analytic and that it posseses an appropriate functional equation (so that the above representation of Γ satisfies the functional equation (2)). For this we need a lemma.

Lemma 5 ('Twisted' $\triangle$ -inequality)**.** *For $s = re^{i\theta}$ and $x \ge 0$ we have*

$$|s+x| \ge (|s| + x)\cos(\theta/2). \tag{12}$$

This gives

$$|s+x| \ge (|s| + x)\sin(\delta/2) \tag{13}$$

when $|\theta| \le \pi - \delta$, $0 < \delta \le \pi$ (since $\cos(\theta/2) \ge \sin(\delta/2)$).

Proof. Using $\cos\theta = 1 - 2\sin^2(\theta/2)$ and $(r+x)^2 \ge 4rx$ we have

$$|s+x|^2 = r^2 + 2rx\cos\theta + x^2 = (r+x)^2 - 4rx\sin^2(\theta/2) \ge (r+x)^2\cos^2(\theta/2)$$

$\square$

Proposition 6. *$\mu(s)$ is analytic in $\mathbb{C}^-$.*

Proof. Define $Q(x) = \frac{1}{2}(x - \lfloor x \rfloor - (x - \lfloor x \rfloor)^2)$. Then $Q(t)$ is an antiderivative of $-P_1(x)$ (so continuous) and $0 \le Q(x) \le 1/8$. We have

$$-\int_\alpha^\beta \frac{P_1(x)}{s+x} dx = \frac{Q(x)}{s+x}\bigg|_\alpha^\beta + \int_\alpha^\beta \frac{Q(x)}{(s+x)^2} dx \tag{14}$$

for $0 < \alpha < \beta < \infty$. Now let $0 < \delta \le \pi$ and $\varepsilon > 0$. Then for $x \ge 0$, $s = re^{i\theta}$ with $r > \varepsilon$ and $|\theta| \le \pi - \delta$ we have (by the above lemma)

$$\left|\frac{Q(x)}{(s+x)^2}\right| \le \frac{|Q(t)|}{\sin^2(\delta/2)(\varepsilon + x)^2} \le \frac{1}{8\sin^2(\delta/2)(\varepsilon + x)^2}.$$

Hence the integral

$$\int_0^\infty \frac{Q(x)}{(s+x)^2} dx$$

is uniformly convergent in compact subsets of $\mathbb{C}^-$ and therefore defines an analytic function by Corollary 1. But by (14) we have

$$\mu(s) = \int_0^\infty \frac{Q(x)}{(s+x)^2}dx. \tag{15}$$

□

Note by (15) and (12), (13) we have

$$|\mu(s)| \leq \frac{1}{8\cos^2(\theta/2)|s|} \tag{16}$$

$$|\mu(s)| \leq \frac{1}{8\sin^2(\delta/2)|s|} \tag{17}$$

for $s = re^{i\theta}$, $|\theta| \leq \pi - \delta$, $0 < \delta \leq \pi$.

Proposition 7. *For $s \in \mathbb{C}^-$ we have*

$$\mu(s) - \mu(s+1) = \left(s + \frac{1}{2}\right)\log\left(\frac{s+1}{s}\right) - 1. \tag{18}$$

Proof. Since $P_1(x+1) = P_1(x)$ we have

$$\begin{aligned}
\mu(s+1) &= -\int_0^\infty \frac{P_1(x)}{(s+1)+x}dx = -\int_0^\infty \frac{P_1(x+1)}{s+x+1}dx \\
&= -\int_1^\infty \frac{P_1(x)}{s+x}dx = \mu(s) - \left(-\int_0^1 \frac{x - \lfloor x \rfloor - 1/2}{s+x}dx\right) \\
&= \mu(s) + \int_0^1 \frac{x - 1/2}{s+x}dx \\
&= \mu(s) + \int_0^1 \left(1 - \frac{s+1/2}{s+x}\right)dx.
\end{aligned}$$

□

Theorem 3 (Complex Stirling's Formula)**.** *For $s \in \mathbb{C}^-$ we have*

$$\Gamma(s) = \sqrt{2\pi}s^{s-\frac{1}{2}}e^{-s}e^{\mu(s)}. \tag{19}$$

Proof. By Proposition 6 the function

$$F(s) = s^{s-\frac{1}{2}}e^{-s}e^{\mu(s)}$$

is analytic in $\mathbb{C}^-$ (we define $s^{s-\frac{1}{2}} = e^{(s-\frac{1}{2})\log s}$ where log is the principal branch of the logarithm). By Proposition 7 we have

$$F(s+1) = (s+1)^{s+1/2}e^{-s-1}e^{\mu(s)-\left(s+\frac{1}{2}\right)\log\left(\frac{s+1}{s}\right)+1} = s^{s+1/2}e^{-s}e^{\mu(s)} = sF(s).$$

Also $F(s)$ is bounded in the region $\mathcal{A} = \{s \in \mathbb{C} : \sigma > 0\}$: Clearly, $e^{\mu(s)}$ is bounded by (16). Writing $s = \sigma + it = |s|e^{i\theta} \in \mathbb{C}$ we have $|s^{s-\frac{1}{2}}e^{-s}| = |s|^{\sigma-1/2}e^{-\theta t}e^{-\sigma}$. Then for $s \in \mathcal{A}$ with $|t| \geq 2$ we have $\sigma - 1/2 \leq 2$, $|s| \leq 2t$ and $-t\theta \leq -\pi|t|/2$. Hence, in this region we have $|s^{s-\frac{1}{2}}e^{-s}| \leq 4t^2 e^{-\pi|t|/2}e^{-1} \to 0$ as $|t| \to \infty$. Hence $F(s)$ is bounded in $\mathcal{A}$. By the Uniqueness Theorem we must have

$$\Gamma(s) = as^{s-\frac{1}{2}}e^{-s}e^{\mu(s)}$$

for some a. Substituting this into the duplication formula (11) gives

$$\begin{aligned} a^2 s^{s-1/2}e^{-s}e^{\mu(s)}(s+1/2)^s e^{-s-1/2}e^{\mu(s+1/2)} &= a2^{1-2s}\sqrt{\pi}(2s)^{2s-1/2}e^{-2s}e^{\mu(2s)} \\ &= a\sqrt{2\pi}s^{2s-1/2}e^{-2s}e^{\mu(2s)}. \end{aligned}$$

Hence

$$a(1+1/2s)^s e^{\mu(s)+\mu(s+1/2)} = \sqrt{2\pi e}e^{\mu(2s)}.$$

Now let s be real and approach infinity. By (16) the exponentials tend to 1 and since $\lim_{x\to\infty}(1+1/2x)^x = \sqrt{e}$ we have $a = \sqrt{2\pi}$. □

This result immediately gives the original form of stirling's formula

Corollary 3. *As $x \to \infty$ in $\mathbb{R}$ we have $\Gamma(x) \sim \sqrt{2\pi}x^{x-1/2}e^{-x}$.*

Another consequence is the following.

Corollary 4 (Rapid decay in vertical strips)**.** *Let $\sigma \in \mathbb{R}$ be fixed. Then as $|t| \to \infty$ we have*

$$|\Gamma(\sigma+it)| \sim \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2}. \tag{20}$$

Proof. Assume $t \geq 0$. By (19) and (16) we have

$$\begin{aligned} \log|\Gamma(\sigma+it)| &= \Re(\log(\Gamma(\sigma+it))) \\ &= \Re((\sigma+it-1/2)\log(\sigma+it)-\sigma-it)+\frac{1}{2}\log 2\pi + O(1/t) \\ &= \Re((\sigma+it-1/2)(\log(\sigma^2+t^2)/2+i\arg(\sigma+it))) \\ &\quad -\sigma+\frac{1}{2}\log 2\pi + O(1/t) \\ &= (\sigma-1/2)\left(\log\left(|t|\sqrt{1+\sigma^2/t^2}\right)\right) - t\arg(\sigma+it) \\ &\quad -\sigma+\frac{1}{2}\log 2\pi + O(1/t) \\ &= (\sigma-1/2)(\log|t|+o(1)) - t(\pi/2-\tan^{-1}(\sigma/t)) \\ &\quad -\sigma+\frac{1}{2}\log 2\pi + o(1) \\ &= (\sigma-1/2)\log|t| - \pi t/2 + \frac{1}{2}\log 2\pi + o(1). \end{aligned}$$

Doing similar stuff for $t \leq 0$ gives the result. □