DIRICHLET CHARACTERS AND PRIMES IN ARITHMETIC PROGRESSIONS

We plan to prove the following.

Theorem 1 (Dirichlet’s Theorem). Let \((a, k) = 1\). Then the arithmetic progression 
\[ \{a + nk : n = 0, 1, 2, \ldots\} = \{m : m \equiv a \mod k\} \]
contains infinitely many primes.

Euler gave an analytic proof of the infinitude of primes by showing that
\[ \sum_{p \leq x} \frac{1}{p} \to \infty. \]
In doing so he used properties of the zeta function
\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}. \]

We will attempt something similar and show that
\[ \sum_{p \leq x, p \equiv a \mod k} \frac{1}{p} \to \infty \]
(for technical reasons we will in fact demonstrate the divergence of \(\sum_{p \equiv a \mod k} \frac{\log p}{p}\)).

Note that the above sum can be written as \(\sum_p f_{a,k}(p)/p\) where \(f_{a,k}\) is the characteristic function of the property \(p \equiv a \mod k\). For example, one could take \(f_{a,k}(p) = \lfloor |\cos(\pi(p - a)/k)| \rfloor\) but this is superficial and no good to anybody. It would be nice if our function, or some slight variant, made sense as the coefficients of a Dirichlet series and allowed for an Euler product. This would already imply more depth than our previous example for \(f_{a,k}\).

1. Dirichlet characters

Definition 1. A Dirichlet Character modulo \(k\) is an arithmetic function \(\chi : \mathbb{N} \to \mathbb{C}\) satisfying

1. \(\chi(n + k) = \chi(n) \ \forall n \in \mathbb{N}\)
2. \(\chi(mn) = \chi(m)\chi(n) \ \forall m, n \in \mathbb{N}\)
3. \(\chi(n) \neq 0 \iff (n, k) = 1.\)
Definition 2. The principal character modulo $k$ is the unique Dirichlet character $\chi_1$ such that $\chi_1(n) = 1 \iff (n, k) = 1$.

Firstly, note that $\chi(1) = 1$ since for $(n, k) = 1$ we have $0 \neq \chi(n) = \chi(1 \cdot n) = \chi(1)\chi(n)$. Also, note that $\chi$’s non-zero values are $\phi(k)$th-roots of unity since for $(n, k) = 1$, we have $n^{\phi(k)} \equiv 1 \mod k$ and hence $\chi(n)^{\phi(k)} = \chi(n^{\phi(k)}) = \chi(1 + mk) = \chi(1) = 1$.

Examples

1. $k = 1$: One character, $\chi(n) = 1 \forall n$.
2. $k = 2$: One character, $\chi(n) = 1$ for $n$ odd, $\chi(n) = 0$ for $n$ even.
3. $k = 3$: We have $\chi_1$. Suppose $\chi \neq \chi_1$. Then $\chi(2) \neq 1$ and $\chi(2)^2 = \chi(4) = \chi(1) = 1$ hence $\chi(2) = -1$.
4. $k = 4$: Similar to $k = 3$.

Note that by periodicity we only need consider one representative $n$ from a given residue class $\bar{n} \mod k$. Also, we may restrict attention to those representatives $n$ with $(n, k) = 1$ since $\chi(n) = 0$ otherwise. In other words, we can restrict our attention to the group $(\mathbb{Z}/k\mathbb{Z})^\times$ of units modulo $k$. By their multiplicative property, we see that any given Dirichlet character $\chi : \mathbb{N} \to \mathbb{C}$ induces a homomorphism $\chi^* : (\mathbb{Z}/k\mathbb{Z})^\times \to \mathbb{C}^\times = GL_1(\mathbb{C})$, $\bar{n} \mapsto \chi(n)$. Conversely, given a homomorphism $f : (\mathbb{Z}/k\mathbb{Z})^\times \to \mathbb{C}^\times$ we can acquire a Dirichlet character by setting $\chi(n) = f(\bar{n})$ when $(n, k) = 1$ and $\chi(n) = 0$ when $(n, k) > 1$.

If we view the set of Dirichlet characters mod $k$ as homomorphisms from $(\mathbb{Z}/k\mathbb{Z})^\times \to \mathbb{C}^\times$ then they form a group when equipped with the pointwise multiplication defined by $(\chi \cdot \chi')(n) = \chi(n)\chi'(n)$. The identity element is given by $\chi_1$ and the inverse of an element $\chi$ is given by $\overline{\chi}$ since $(\chi \cdot \overline{\chi})(n) = |\chi(n)|^2 = 1 = \chi_1(n)$. This means that the Dirichlet characters modulo $k$ can be thought of as the elements of something called the dual group $\hat{G}$ of the group $G = (\mathbb{Z}/k\mathbb{Z})^\times$. A famous result of harmonic analysis (Pontryagin Duality) implies that $G$ is isomorphic to $\hat{G}$, and hence we have the first part of the following theorem.

**Theorem 2.** There are exactly $\phi(k)$ Dirichlet characters modulo $k$. Also, for any given $n$ coprime to $k$ with $n \not\equiv 1 \mod k$ there exists a $\chi$ such that $\chi(n) \neq 1$.

**Theorem 3 (Orthogonality).** Let $k$ be a positive integer and let $\chi$ be a Dirichlet character modulo $k$. Then

\[
\sum_{n=1}^{k} \chi(n) = \begin{cases} 
\phi(k) & \text{if } \chi = \chi_1 \\
0 & \text{otherwise}
\end{cases}.
\]
Let \( n \) be a positive integer. Then

\[
\sum_{\chi \mod k} \chi(n) = \begin{cases} 
\phi(k) & \text{if } n \equiv 1 \mod k, \\
0 & \text{otherwise},
\end{cases}
\]

where the summation is over all Dirichlet characters mod \( k \). Let \( \chi, \chi' \) be Dirichlet characters mod \( k \). Then

\[
\sum_{n=1}^{k} \chi(n)\overline{\chi'}(n) = \begin{cases} 
\phi(k) & \text{if } \chi = \chi', \\
0 & \text{otherwise}.
\end{cases}
\]

For positive integers \( n_1, n_2 \) we have

\[
\sum_{\chi \mod k} \chi(n_1)\overline{\chi}(n_2) = \begin{cases} 
\phi(k) & \text{if } n_1 \equiv n_2 \mod k \text{ and } (n_1, k) = 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Proof. For the first sum the result is obvious if \( \chi = \chi_1 \). If \( \chi \neq \chi_1 \) then pick an \( m \) with \( (m, k) = 1 \) such that \( \chi(m) \neq 1 \). Then as \( n \) runs through a reduced residue system mod \( k \) so does \( mn \). Therefore,

\[
\chi(m) \sum_{n=1}^{k} \chi(n) = \sum_{n=1}^{k} \chi(mn) = \sum_{n=1}^{k} \chi(n)
\]

and the result follows.

For the second sum note that if \( (n, k) > 1 \) then all terms in the sum are zero. If \( n \equiv 1 \mod k \) then \( \chi(n) = \chi(1) = 1 \) for all characters and so the sum simply equals the number of characters mod \( k \), which is \( \phi(k) \) by Theorem 2. So now suppose that \( (n, k) = 1 \) and that \( n \neq 1 \mod k \). Pick a \( \chi' \) such that \( \chi'(n) \neq 1 \). Then as \( \chi \) runs over the characters mod \( k \) so does \( \chi' \chi \), since they form a group. Therefore,

\[
\chi'(n) \sum_{\chi} \chi(n) = \sum_{\chi} \chi'(n)\chi(n) = \sum_{\chi} (\chi' \cdot \chi)(n) = \sum_{\chi} \chi(n)
\]

and the sum is therefore equal to zero in this case.

For the third sum take \( \chi = \chi\overline{\chi}' \) in (1). Finally, for the fourth sum note that if either \( (n_1, k) \) or \( (n_2, k) \) is \( > 1 \) then the sum is zero. So suppose \( (n_1, k) = (n_2, k) = 1 \) and let \( \overline{n_2} \) denote the inverse of \( n_2 \) modulo \( k \) i.e. the number satisfying \( n_2\overline{n_2} \equiv 1 \mod k \). We now apply (2) with \( n = n_1\overline{n_2} \). The sum in question is given by

\[
\sum_{\chi} \chi(n) = \sum_{\chi} \chi(n_1)\chi(\overline{n_2}) = \sum_{\chi} \chi(n_1)\overline{\chi}(n_2).
\]

This last equality follows on noting that \( \chi(n_1)\chi(\overline{n_2}) = \chi(n_2\overline{n_2}) = \chi(1) = 1 \) and hence \( \chi(\overline{n_2}) = \overline{\chi(n_2)} \) since \( \chi \) maps onto the unit circle for integers coprime to \( k \). If \( n_1 \equiv n_2 \)
mod \( k \) then \( n \equiv 1 \mod k \) and the above sum equals \( \phi(k) \) by (2). If \( n_1 \not\equiv n_2 \mod k \) then \( n \not\equiv 1 \mod k \) and the remaining case follows.

2. Dirichlet \( L \)-functions

Let \( \chi \) be a Dirichlet character mod \( k \). Then the Dirichlet \( L \)-function associated to \( \chi \) is given by

\[
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad s = \sigma + it
\]

\textbf{Proposition 2.1.} The series for \( L(s, \chi) \) converges absolutely for \( \sigma > 1 \) and for fixed \( \delta > 0 \) it converges uniformly for \( \sigma \geq 1 + \delta \). It is therefore analytic in the region \( \sigma > 1 \).

\textit{Proof.} Postponed.

The derivative of \( L(s, \chi) \) is given by the series

\[
L'(s, \chi) = -\sum_{n=1}^{\infty} \frac{\chi(n) \log n}{n^s}.
\]

\textbf{Proposition 2.2.} If \( \chi \neq \chi_1 \) then \( L(s, \chi) \) and \( L'(s, \chi) \) converge (conditionally) for \( \sigma > 0 \). In particular, their values at \( s = 1 \) are defined.

\textit{Proof.} If \( \chi \neq \chi_1 \) then by partial summation we have

\[
\sum_{n \leq x} \frac{\chi(n)}{n^s} = \frac{1}{x^s} \sum_{n \leq x} \chi(n) + s \int_{1}^{x} \left[ \sum_{n \leq t} \chi(n) \right] t^{-s-1} dt.
\]

By (1) the sums \( \sum_{n \leq x} \chi(n) \) are bounded. Therefore

\[
\sum_{n \leq x} \frac{\chi(n)}{n^s} \ll x^{-\sigma} + s \int_{1}^{x} t^{-\sigma-1} dt.
\]

This last expression is \( O(1) \) for \( \sigma > 0 \) and so the result follows on letting \( x \to \infty \). For the result involving the derivative just use \( \log t \ll t^\epsilon \) in the above.

The following is not absolutely necessary for our purposes but it does give more meaning to some of our results.

\textbf{Theorem 4 (Euler product).} For \( \sigma > 1 \) we have the absolutely convergent product

\[
L(s, \chi) = \prod_{p} \left( 1 - \frac{\chi(p)}{p^s} \right)^{-1}.
\]
Also, for $\sigma > 1$ we have

\[(10) \quad \frac{1}{L(s, \chi)} = \sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s}\]

and the sum is absolutely convergent in this region.

**Proof.** Note the factors in the product are the sums of geometric series:

\[
\left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = \sum_{m=0}^{\infty} \frac{\chi(p^m)}{p^{ms}}.
\]

Taking the product over primes less than a given $x$ gives

\[
\prod_{p \leq x} \sum_{m=0}^{\infty} \frac{\chi(p^m)}{p^{ms}} = \sum_{p_1, \ldots, p_j \leq x} \frac{\chi(p_1^{m_1} \cdots p_j^{m_j})}{(p_1^{m_1} \cdots p_j^{m_j})^s} = \sum_{n \in A(x)} \frac{\chi(n)}{n^s}
\]

where $A(x) = \{n \in \mathbb{N} : p|n \implies p \leq x\}$. Clearly, $\lim_{x \to \infty} A(x) = \mathbb{N}$. Now,

\[
L(s, \chi) - \sum_{n \in A(x)} \frac{\chi(n)}{n^s} \ll \sum_{n > x} \frac{\chi(n)}{n^s} \to 0
\]

as $x \to \infty$ since the series $\sum_{n=1}^{\infty} \chi(n)n^{-s}$ is convergent for $\sigma > 1$.

Since an absolutely convergent product of non-zero terms is non-zero we see that $L(s, \chi) \neq 0$ for $\sigma > 1$. Taking the reciprocal gives

\[
\frac{1}{L(s, \chi)} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right).
\]

Expanding this product we see that the resultant series has coefficients $\mu(n)\chi(n)$ and so (10) follows. If the rigour police show up then just use the previous argument. Absolute convergence of the series follows on applying the integral test after using $|\mu(n)\chi(n)| \leq 1$ and $|n^s| = n^\sigma$.

\[\square\]

3. **Dirichlet’s Theorem**

We are now in a position to prove Dirichlet’s Theorem. We shall in fact prove the following

**Theorem 5.** Suppose $(a, k) = 1$. Then

\[(11) \quad \sum_{\substack{p \leq x \\ p \equiv a \mod k}} \frac{\log p}{p} = \frac{1}{\phi(k)} \log x + O(1).\]

The starting point is to use (4) as a suitable characteristic function.
Lemma 3.1. Let $(a, k) = 1$. Then

$$\sum_{p \leq x, \ p \equiv a \mod k} \frac{\log p}{p} = \frac{1}{\phi(k)} \log x + \frac{1}{\phi(k)} \sum_{\chi \neq \chi_1} \chi(a) \sum_{p \leq x} \frac{\chi(p) \log p}{p} + O(1)$$

Proof. By (4) we have

$$\sum_{p \leq x, \ p \equiv a \mod k} \frac{\log p}{p} = \sum_{p \leq x} \frac{\log p}{p} \left[ \frac{1}{\phi(k)} \sum_{\chi} \bar{\chi}(a) \chi(p) \right]$$

$$= \frac{1}{\phi(k)} \sum_{p \leq x} \frac{\log p}{p} \left[ \chi_1(p) + \sum_{\chi \neq \chi_1} \bar{\chi}(a) \chi(p) \right]$$

$$= \frac{1}{\phi(k)} \sum_{p \leq x} \chi_1(p) \log p + \frac{1}{\phi(k)} \sum_{\chi \neq \chi_1} \bar{\chi}(a) \sum_{p \leq x} \frac{\chi(p) \log p}{p}.$$

Now,

$$\sum_{p \leq x} \frac{\chi_1(p) \log p}{p} = \sum_{p \leq x, \ (p, k) = 1} \frac{\log p}{p} = \left[ \sum_{p \leq x} - \sum_{p \leq x, \ p \mid k} \right] \frac{\log p}{p}$$

$$= \sum_{p \leq x, \ (p, k) = 1} \frac{\log p}{p} + O(1)$$

$$= \log x + O(1).$$

The above Lemma implies that if we can show

$$\sum_{p \leq x} \frac{\chi(p) \log p}{p} = O(1)$$

for $\chi \neq \chi_1$, then Dirichlet’s Theorem will follow. This is now our main focus.

**Informal Discussion**: As we have seen before, incorporating prime powers into a sums involving $\log p$ usually just adds an $O(1)$ term. So we can expect

$$\sum_{p \leq x} \frac{\chi(p) \log p}{p} = \sum_{n \leq x} \frac{\chi(n) \Lambda(n)}{n} + O(1)$$

where $\Lambda(n) = \log p$ if $n = p^m$ and $\Lambda(n) = 0$ otherwise. Why are we always adding in prime power terms if it doesn’t really change anything? Well, this latter sum admits a more concise description as the partial sums of the series for $-L'(1, \chi)/L(1, \chi)$. 
Indeed, on logarithmic differentiation of the Euler product: 
\[ L(s, \chi) = \prod_p (1 - \chi(p) p^{-s})^{-1}, \]
we have
\[
- \frac{L'(s, \chi)}{L(s, \chi)} = \sum_p \frac{\log p}{1 - \chi(p) p^{-s}} = \sum_p \sum_{m=0}^\infty (\log p) \chi(p^m) p^{-ms} = \sum_{n=1}^\infty \frac{\chi(n) \Lambda(n)}{n^s}.
\]

From this we see that
\[
\sum_{p \leq x} \frac{\chi(p) \log p}{p} \approx - \frac{L'(1, \chi)}{L(1, \chi)}
\]
and this is bounded as long as \( L(1, \chi) \neq 0 \). We now attempt to make our discussion into a more rigorous argument. Our first step is to demonstrate a relationship similar to (16), but in a more quantitative form. We then show that this form is bounded if \( L(1, \chi) \neq 0 \) for \( \chi \neq \chi_1 \). Finally, we prove the latter.

**Lemma 3.2.** We have
\[
\sum_{p \leq x} \frac{\chi(p) \log p}{p} = \sum_{n \leq x} \frac{\chi(n) \Lambda(n)}{n} + O(1).
\]

**Proof.** We have
\[
\sum_{n \leq x} \frac{\chi(n) \Lambda(n)}{n} = \sum_{p \leq x} \frac{\chi(p^m) \log p}{p^m} = \sum_{p \leq x} \frac{\chi(p) \log p}{p} + \sum_{p^m \leq x, m \geq 2} \frac{\chi(p^m) \log p}{p^m}.
\]
Now, the second sum in the above is \( \ll \) than
\[
\sum_p \log p \sum_{m \geq 2} \frac{1}{p^m} = \sum_p \frac{\log p}{p(p-1)} \ll \sum_n \frac{\log n}{n(n-1)} = O(1).
\]

**Lemma 3.3.** For \( \chi \neq \chi_1 \) we have
\[
\sum_{p \leq x} \frac{\chi(p) \log p}{p} = -L'(1, \chi) \sum_{n \leq x} \frac{\mu(n) \chi(n)}{n} + O(1).
\]
Proof. By the previous Lemma and the fact that $\Lambda(n) = \sum_{d|n} \mu(d) \log(n/d)$ we have

$$\sum_{p \leq x} \frac{\chi(p) \log p}{p} + O(1) = \sum_{n \leq x} \frac{\chi(n)\Lambda(n)}{n} = \sum_{n \leq x} \frac{\chi(n)}{n} \sum_{d|n} \mu(d) \log(n/d)$$

(20)

$$= \sum_{d \leq x} \frac{\chi(d)\mu(d)}{d} \sum_{m \leq x/d} \frac{\chi(m) \log m}{m}$$

after rearranging in terms of the divisors $d$ and using the multiplicative properties of $\chi$. Now,

$$-L'(1, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m) \log m}{m} = \left[ \sum_{m \leq Y} \sum_{m \geq Y} \frac{\chi(m) \log m}{m} \right],$$

and

$$\sum_{m \geq Y} \frac{\chi(m) \log m}{m} = -\left[ \sum_{n \leq Y} \frac{\chi(n) \log Y}{Y} - \int_{Y}^{\infty} \left[ \sum_{n \leq t} \frac{\chi(n)}{n} \right] \left( \log t \right)' \frac{dt}{t} \right] \ll \frac{\log Y}{Y}.$$

Here we have used the fact that $\chi \neq \chi_1$ and hence the sum $\sum_{n \leq Y} \chi(n)$ is bounded. Therefore,

$$\sum_{n \leq x} \frac{\chi(n)\Lambda(n)}{n} = -L'(1, \chi) \sum_{d \leq x} \frac{\chi(d)\mu(d)}{d} + O\left( \sum_{d \leq x} \frac{1}{d} \log x/d \right)$$

(21)

$$= -L'(1, \chi) \sum_{d \leq x} \frac{\chi(d)\mu(d)}{d} + O(1)$$

since

$$\sum_{d \leq x} (\log x - \log d) = \lfloor x \rfloor \log x - x \log x + O(x) = O(x).$$

Note that by equation (10), the sum

$$\sum_{n \leq x} \frac{\mu(n)\chi(n)}{n}$$

looks a lot like $1/L(1, \chi)$ and so we have essentially established the relation (16).

Lemma 3.4. Suppose $\chi \neq \chi_1$. If $L(1, \chi) \neq 0$ then

$$\sum_{n \leq x} \frac{\mu(n)\chi(n)}{n} = O(1).$$

(22)
Proof. Note that
\[
S := \sum_{n \leq x} \frac{\mu(n) \chi(n)}{n} \sum_{m \leq x/n} \frac{\chi(m)}{m} = \sum_{m,n \leq x} \frac{\chi(mn) \mu(n)}{mn}
\]
(23) after rearranging. We now group together the terms \(mn\), writing \(mn = \ell\) say, and rearrange to give
\[
S = \sum_{\ell \leq x} \frac{\chi(\ell)}{\ell} \sum_{n|\ell} \mu(n) = 1
\]
(24) after using
\[
\sum_{n|\ell} \mu(n) = \begin{cases} 1 & \text{if } \ell = 1, \\ 0 & \text{otherwise.} \end{cases}
\]
On the other hand,
\[
S = \sum_{n \leq x} \frac{\mu(n) \chi(n)}{n} \left[ L(1, \chi) - \sum_{m > x/n} \frac{\chi(m)}{m} \right]
= \sum_{n \leq x} \frac{\mu(n) \chi(n)}{n} \left[ L(1, \chi) + O(n/x) \right]
\]
(25)
\[
= L(1, \chi) \sum_{n \leq x} \frac{\mu(n) \chi(n)}{n} + O \left( \frac{1}{x} \sum_{n \leq x} \chi(n) \mu(n) \right)
= L(1, \chi) \sum_{n \leq x} \frac{\mu(n) \chi(n)}{n} + O \left( \frac{1}{x} \sum_{n \leq x} \chi(n) \mu(n) \right)
\]
Since \(L(1, \chi) \neq 0\), we may divide through and the result follows. \(\square\)

4. The non-vanishing of \(L(1, \chi)\) for non-principal characters.

As the section heading suggests, we plan to prove the following.

**Theorem 6.** If \(\chi \neq \chi_1\) then \(L(1, \chi) \neq 0\).

The proof is split into two cases: one where \(\chi\) is real, and the other where \(\chi\) is complex. We will start with the real case.

**Lemma 4.1.** Suppose \(\chi\) is a real Dirichlet character mod \(k\). Let
\[
A(n) = \sum_{d|n} \chi(d).
\]
(26)
Then $A(n) \geq 0$ for all $n$ and $A(n) \geq 1$ if $n$ is a square.

**Proof.** First note $A(n)$ is multiplicative since it is the convolution of two multiplicative functions, namely $1(n) := 1 \forall n$ and $\chi(n)$. Therefore, it is determined by its values at prime powers. We have

\[
\begin{align*}
\chi(p) = 0 & \implies A(p^\alpha) = 1, \\
\chi(p) = 1 & \implies A(p^\alpha) = d(p^\alpha) = \alpha + 1, \\
\chi(p) = -1 & \implies A(p^\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is even}, \\
0 & \text{if } \alpha \text{ is odd}.
\end{cases}
\end{align*}
\]

Writing $n = \prod_p p^{\alpha_p}$ we see $A(n) = \prod_p A(p^{\alpha_p})$ and hence $A(n) \geq 0$ for all $n$. If $n$ is square then all the powers $\alpha_p$ in the prime decomposition of $n$ are even and so $A(n) \geq 1$. \hfill \square

**Proposition 4.2.** Suppose $\chi$ is a real non-principal Dirichlet character mod $k$. Let

\[ B(x) = \sum_{n \leq x} \frac{A(n)}{n^{1/2}}. \tag{27} \]

Then

1. $B(x) \to \infty$ as $x \to \infty$.
2. $B(x) = 2x^{1/2}L(1, \chi) + O(1)$ and hence $L(1, \chi) \neq 0$.

**Proof.** For the first part note that by the Lemma

\[
B(x) \geq \sum_{n \leq x} \frac{A(n)}{n^{1/2}} \geq \sum_{n \leq x} \frac{1}{n^{1/2}} = \sum_{m \leq x^{1/2}} \frac{1}{m} \sim \frac{1}{2} \log x \to \infty.
\]

For the second part we have

\[
B(x) = \sum_{n \leq x} \frac{1}{n^{1/2}} \sum_{d|n} \chi(d) = \sum_{d \leq x} \frac{\chi(d)}{d^{1/2}} \sum_{m \leq x/d} \frac{1}{m^{1/2}}, \text{ after rearranging}
\]

\[
= \sum_{d \leq x} \frac{\chi(d)}{d^{1/2}} \left[ 2 \left( \frac{x}{d} \right)^{1/2} + O(1) \right], \text{ using Euler summation}
\]

\[= 2x^{1/2} \sum_{d \leq x} \frac{\chi(d)}{d} + O\left( \sum_{d \leq x} \chi(d)d^{-1/2} \right)
\]

\[= 2x^{1/2} \left( L(1, \chi) - \sum_{d > x} \frac{\chi(d)}{d} \right) + O(1)
\]

\[= 2x^{1/2} L(1, \chi) + O(1) \tag{28}
\]
where we have used

\[ \sum_{d > x} \frac{\chi(d)}{d} = -\frac{1}{x} \sum_{d \leq x} \chi(d) + \int_{x}^{\infty} \sum_{d \leq t} \chi(d) \frac{1}{t^2} dt = O(x^{-1}). \]

We can now turn to the case where \( \chi \) is complex. We will proceed towards a contradiction.

**Lemma 4.3.** If \( \chi \neq \chi_1 \) and \( L(1, \chi) = 0 \) then

(29) \[ L'(1, \chi) \sum_{n \leq x} \frac{\mu(n) \chi(n)}{n} = \log x + O(1). \]

**Proof.** We have

\[ S := \sum_{n \leq x} \frac{\mu(n) \chi(n)}{n} \sum_{m \leq x/n} \frac{\chi(m) \log(x/mn)}{m} \]

(30) \[ = \sum_{m,n \leq x} \frac{\mu(n) \chi(mn) \log(x/mn)}{mn}. \]

Once again, we group together the terms \( mn \), writing \( mn = \ell \) say, and rearrange to give

(31) \[ S = \sum_{\ell \leq x} \frac{\chi(\ell) \log(x/\ell)}{\ell} \sum_{n | \ell} \mu(n) = \log x \]

after using

\[ \sum_{n | \ell} \mu(n) = \begin{cases} 1 & \text{if } \ell = 1, \\ 0 & \text{otherwise}. \end{cases} \]
On the other hand,

\[
S = \sum_{n \leq x} \frac{\mu(n) \chi(n)}{n} \left[ \sum_{m \leq x/n} \frac{\chi(m)}{m} (-\log m + \log(x/n)) \right]
\]

\[
= \sum_{n \leq x} \frac{\mu(n) \chi(n)}{n} \left[ -\sum_{m \leq x/n} \frac{\chi(m) \log m}{m} + \log(x/n) \sum_{m \leq x/n} \frac{\chi(m)}{m} \right]
\]

\[
= \sum_{n \leq x} \frac{\mu(n) \chi(n)}{n} \left[ \left( L'(1, \chi) + \sum_{m > x/n} \frac{\chi(m) \log m}{m} \right) \right.
\]

\[
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \log(x/n) \left( L(1, \chi) - \sum_{m > x/n} \frac{\chi(m)}{m} \right) \bigg]
\]

\[
= \sum_{n \leq x} \frac{\mu(n) \chi(n)}{n} \left[ L'(1, \chi) + O \left( \frac{\log(x/n)}{x/n} \right) \right.
\]

\[
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \log(x/n) \left( L(1, \chi) - \sum_{m > x/n} \frac{\chi(m)}{m} \right) \bigg].
\]  

(32)

Since we’re assuming \( L(1, \chi) = 0 \) this equals

\[
L'(1, \chi) \sum_{n \leq x} \frac{\mu(n) \chi(n)}{n} + O \left( \frac{1}{x} \sum_{n \leq x} \log(x/n) \right).
\]  

(33)

The result now follows on noting that the sum in ‘big O’ term is \( \ll x \), as has been seen previously. \( \square \)

Let

\[
S(k) = \left\{ \chi \mod k : \chi \neq \chi_1, \ L(1, \chi) = 0 \right\}
\]

(34)

and let

\[
N(k) = |S(k)|.
\]  

(35)

Now, if \( L(1, \chi) = 0 \) then \( L(1, \overline{\chi}) = 0 \) and so if \( \chi \in S(k) \) then \( \overline{\chi} \in S(k) \) (note these characters are distinct since \( \chi \neq \overline{\chi} \) for complex characters). Therefore, \( N(k) \) is even.

**Proposition 4.4.** We have

\[
\sum_{\substack{p \leq x \\backslash \\phi(k)}} \frac{\log p}{p} = \frac{1 - N(k)}{\phi(k)} \log x + O(1).
\]  

(36)
This implies that \( N(k) = 0 \) since otherwise \( N(k) \geq 2 \) and hence the right hand side of the above would be negative for large enough \( x \), contrary to the sum on the left being made up solely of positive terms.

**Proof.** By Lemmas 3.1 and 3.3 we have

\[
\sum_{\substack{p \leq x \\
p \equiv 1(k)}} \frac{\log p}{p} = \frac{1}{\phi(k)} \log x + \frac{1}{\phi(k)} \sum_{\chi \neq 1} \sum_{p \leq x} \frac{\chi(p) \log p}{p} + O(1)
\]

\begin{equation}
= \frac{1}{\phi(k)} \log x - \frac{1}{\phi(k)} \sum_{\chi \neq 1} L'(1, \chi) \sum_{n \leq x} \frac{\mu(n) \chi(n)}{n} + O(1).
\end{equation}

(37)

Now, by Lemma 3.4 the sum over \( n \leq x \) is \( O(1) \) if \( L(1, \chi) \neq 0 \). If \( L(1, \chi) = 0 \) then the sum times \( L'(1, \chi) \) is \( \log x + O(1) \) by Lemma 4.3. Therefore, after writing \( 1_P \) for the characteristic function of a given property \( P \), we have

\[
\sum_{\substack{p \leq x \\
p \equiv 1(k)}} \frac{\log p}{p} = \frac{1}{\phi(k)} \log x - \frac{1}{\phi(k)} \sum_{\chi \neq 1} \left[ 1_{L(1, \chi) \neq 0} \cdot O(1) \right] + \left[ 1_{L(1, \chi) = 0} \cdot \left( \log x + O(1) \right) \right] + O(1)
\]

\begin{equation}
= \frac{1}{\phi(k)} \log x - \frac{N(k)}{\phi(k)} \log x + O(1).
\end{equation}

(38)

\( \square \)