1.1.1. Proposition. *The inequalities*

$$\log(2)x - \log(4x) \le \psi(x) \le 2\log(2)x + \frac{\log^2(x)}{\log(2)}$$

hold for $x \ge 1$*.*

Proof. The terms in the last sum in the computation

$$\begin{aligned} T(x) - 2T\left(\frac{x}{2}\right) &= \sum_{n\le x} \log(n) - 2\sum_{m\le x/2} \log(m) \\ &= \sum_{n\le x} \log(n) - 2\sum_{2m\le x} \log(2m) + 2\sum_{2m\le x} \log(2) \\ &= \sum_{n\le x} (-1)^{n-1}\log(n) + 2\left[\frac{x}{2}\right]\log(2) \end{aligned}$$

alternate in sign and increase in magnitude. So

$$\left|T(x) - 2\,T\left(\frac{x}{2}\right) - 2\left[\frac{x}{2}\right]\log(2)\right| \le \log([x])$$

for $x \ge 1$. Thus

$$\log(2)x - \log(4x) \le T(x) - 2\,T\left(\frac{x}{2}\right) \le \log(2)x + \log(x).$$

Substituting the expression

$$T(x) = \sum_{n\le x} \psi\left(\frac{x}{n}\right)$$

into $T(x) - 2\,T(x/2)$ yields

$$\psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) - \cdots = T(x) - 2\,T\left(\frac{x}{2}\right).$$

Then

$$\psi(x) \ge \log(2)x - \log(4x)$$

since ψ is an increasing and nonnegative function. And

$$\psi(x) - \psi\left(\frac{x}{2}\right) \le \log(2)x + \log(x)$$

for the same reason. Adding up the inequalities

$$\psi\left(\frac{x}{2^j}\right) - \psi\left(\frac{x}{2^{j+1}}\right) \le \log(2)2^{-j}x + \log\left(\frac{x}{2^j}\right)$$

for $j = 0, 1, 2, \ldots, [\log(x)/\log(2)] - 1$ yields

$$\psi(x) \le 2\log(2)x + \left[\frac{\log(x)}{\log(2)}\right]\log(x) \le 2\log(2)x + \frac{\log^2(x)}{\log(2)}$$

since $\psi(x/2^{j+1}) = 0$ when $x/2^{j+1} < 2$. □

1.2.1. Bertrand's Postulate. *For every $x \geq 2$ there exists at least one prime p with $x/2 < p \leq x$.*

Proof. For $2 \leq x \leq 797$ the interval $(x/2, x]$ contains a prime by a trick of E. G. H. Landau: The chain $2, 3, 5, 7, 11, 17, 31, 59, 107, 211, 401, 797$ consists of primes and each is smaller than twice its predecessor. To detect primes in the intervals $(x/2, x]$ for $x \geq 797$ we show that the difference $\vartheta(x) - \vartheta(x/2)$ is positive.

Fetch the inequality

$$\psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) \geq T(x) - 2T\left(\frac{x}{2}\right)$$

from the proof of Proposition 1.1.1. Retention of the term $\psi(x/3)$ is an idea due to Ramanujan. Now

$$\begin{aligned}
\vartheta(x) - \vartheta\left(\frac{x}{2}\right) &\geq \psi(x) - 2\psi(x^{1/2}) - \psi\left(\frac{x}{2}\right) \\
&\geq T(x) - 2T\left(\frac{x}{2}\right) - \psi\left(\frac{x}{3}\right) - 2\psi(x^{1/2}) \\
&\geq \log(2)x - \log(4x) - 2\log(2)\frac{x}{3} \\
&\quad - \frac{\log^2(x/3)}{\log(2)} - 4\log(2)x^{1/2} - 2\frac{\log^2(x^{1/2})}{\log(2)} \\
&\geq \frac{\log(2)}{3}x - \log(4x) - \frac{3\log^2(x)}{2\log(2)} - 4\log(2)x^{1/2}
\end{aligned}$$

by Proposition 1.1.1 and its proof. So $\vartheta(x) - \vartheta(x/2) \geq f(x)$ with

$$f(x) = \frac{\log(2)}{3}x - \log(4x) - \frac{3\log^2(x)}{2\log(2)} - 4\log(2)x^{1/2}.$$

The derivative

$$f'(x) = \frac{\log(2)}{3} - \frac{1}{x} - \frac{3\log(x)}{x\log(2)} - \frac{2\log(2)}{x^{1/2}}$$

is an increasing function on $x \geq 797$ since $\log(x)/x$ is decreasing on $x \geq e$. Then $f'(x) > 0$ on $x \geq 797$ because $f'(797) = 0.14$. Thus $f(x)$ is increasing on this interval and hence positive there because $f(797) = 1.2$. □