NTNU
Institutt for matematiske fag

MA2501 Numerical Methods

Exercise set Project 2

Spring 2024

Deadline: Monday, March 22
You may write solutions in Norwegian or English, as preferable. The most important part is *how* you arrive at an answer, not the answer itself.

$\boxed{1}$ Consider the following Newton-Cotes formulas with points $p \in \{1, 2, 4\}$:

$$\text{1-point} \qquad \int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(f_0 + f_1)$$

$$\text{2-point} \qquad \int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}(f_0 + 4f_1 + f_2)$$

$$\text{4-point} \qquad \int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4)$$

where $x_i = x_0 + hi$ for a fixed local mesh size $h = (x_p - x_0)/p$. Consider the composite integral on an arbitrary interval $[a, b]$:

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)\,dx \approx \sum_{k=1}^{n} \sum_{i=0}^{p} D_p h_\delta f(x_{ki}) \tag{1}$$

where $x_{ki} = a + \delta(k-1) + h_\delta i$ is the global node formula, $\delta = (b-a)/n$ is the global mesh size, $h_\delta = \delta/p$ is the local mesh size, and D_p is the scheme dependent constant. Use the Newton-Cotes formulas to compute the following integrals (whose analytical values are known) numerically:

$$\int_0^5 e^{-x/3}(\sin(2\pi x) + 1)\,dx = \frac{3(36\pi^2 + 6\pi + 1)(1 - e^{-5/3})}{36\pi^2 + 1}$$

$$\int_0^6 x^3 \cos(3\pi x)^2\,dx = \frac{3(108\pi^2 + 1)}{2\pi^2}$$

Use $n = \{2^m\}_{m=1}^{6}$ in the composite integral (1) and create a loglog-plot for the approximation errors (one figure for each integral, 5 graphs in each figure). Describe what you observe from the graphs. For this purpose, apply the relative error formula:

$$E = \left| \frac{I_{\text{ex}} - I_{\text{num}}}{I_{\text{ex}}} \right| \tag{2}$$

$\boxed{2}$ This time, we focus on Gauss-Legendre quadrature.

a) The Legendre polynomials are defined by the recursive relation

$$
\begin{aligned}
&L_0(x) = 1 \qquad , \qquad L_1(x) = x \\
&(n+1)L_{n+1}(x) = (2n+1)xL_n(x) - nL_{n-1}(x) \qquad n \geq 1
\end{aligned}
$$

Describe an algorithm with *linear* running time to evaluate $L_n(x)$ at $x \in [-1, 1]$ for $n \geq 0$. Explain how to use it for evaluating $L_n'(x)$ and $L_n''(x)$. What are the analytical formulas for the boundary values $L_n(\pm 1)$, $L_n'(\pm 1)$ and $L_n''(\pm 1)$?

b) If $\{\cos(\theta_{kn})\}_{k=1}^{n}$ are zeros of $L_n(x)$, then

$$
\frac{(2k-1)\pi}{2n+1} < \theta_{kn} < \frac{2k\pi}{2n+1} \qquad , \qquad 1 \leq k \leq n
$$

Derive an initial guess for an iterative solver that can find the zeros of L_n. Describe how you can reduce the total running time with 50% by exploiting the distribution of the zeros on $[-1, 1]$ for any n.

c) Describe how to use Gauss-Legendre quadrature on an arbitrary interval $[a, b]$.

d) Compute the integrals below with Gauss-Legendre quadrature, using $n \in [1, 20]$:

$$
\begin{aligned}
&\int_0^1 x^{20} - 3x^{17} + 7x^{16} - x^{13} + 4x^{11} - 2x^8 + 9x^5 \, dx = \frac{3925}{2142} \\
&\int_0^3 x\cos(x^2)e^{-x^2} \, dx = \frac{1 + (\sin(9) - \cos(9))e^{-9}}{4}
\end{aligned}
$$

This requires a subroutine for computing nodes and weights. Apply Olver's method for the nonlinear equation solving. This cubic scheme is given by

$$
x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{f''(x_k)f(x_k)^2}{2f'(x_k)^3} \tag{3}
$$

Plot the approximation errors in semilog-plots and describe the graphs. Use the relative error formula in (2).