NTNU
Institutt for matematiske fag

MA2501 Numerical Methods

Exercise set 3

Spring 2024

Deadline: Monday 6. April

You may write solutions in Norwegian or English, as preferable. The most important part is *how* you arrive at an answer, not the answer itself.

$\boxed{1}$ In this task, we examine perturbation of second-order nonlinear oscillators, using the classical RK4 integrator. Recall that the Butcher tableau is given by

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

a) Consider the Van der Pol oscillator, defined by the second-order equation

$$x'' - \mu(1 - x^2)x' + x = 0 \tag{1}$$

Transform it into a system with $y = x'$. Use initial condition $(x_0, y_0) = (0, 0.5)$ on the time interval $t \in [0, 30]$ with steps $N = 6000$ and parameter values $\mu = \{0.01, 0.05, 0.1, 0.4, 0.8, 1.5, 2.2, 2.9, 4.0\}$. Plot the solutions in the (t, x)-coordinate system (9 separate figures). Describe what you see when μ changes.

b) Consider the Duffing oscillator, defined by the second-order equation

$$x'' + \mu x' - x + x^3 = \Gamma\cos(\omega t) \tag{2}$$

Transform it into a system with $y = x'$. Use initial condition $(x_0, y_0) = (1, 0)$ on the time interval $t \in [0, 80]$ with steps $N = 80000$ and parameter values $\mu = 0.3$, $\omega = 1.2$, and $\Gamma = \{0.20, 0.25, 0.29, 0.31, 0.34, 0.37, 0.51, 0.57, 0.65\}$. Plot the solutions in the (t, x)-coordinate system (9 separate figures). Describe what you see when Γ changes.

(Hint): The codes for the tasks are almost identical. If the code for 1a) works, you just need to copy, paste, and make some small modifications to get the code for 1b).

$\boxed{2}$ Consider the damped linear oscillator, defined by the second-order equation

$$x'' + 2\mu x' + \lambda^2 x = 0 \tag{3}$$

where the parameters satisfy $\mu, \lambda > 0$ and $d^2 = \lambda^2 - \mu^2 > 0$.

a) Find the analytical solution satisfying $x(0) = 1$ and $x'(0) = -\mu$.

b) Write down the linear equation systems that occur when using Gauss integrators of order 2 and 4, and the numerical solution for each time step. The Butcher tableaus are listed below.

$$\begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array} \quad , \quad \begin{array}{c|cc} \frac{1}{2} - \frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

c) Use these integrators to approximate the solution, using step size $h = \{4/2^i\}_{i=3}^{8}$, $\lambda = \{i+1\}_{i=1}^{3}$, and $\mu = \{0.4j\}_{j=1}^{3}$. Run the simulations until $T = 6$. Visualize the relative error of the simulations in three loglog-plots ($N = 2^i$ on the x-axis). Create one figure for each λ-value, containing both integrators and all μ-values.