MA2501
NUMERICAL METHODS
NTNU, SPRING 2023

Exercise Set 2

Exercise sets are to be handed in individually by each student using the OVSYS system. Each set will be graded (godkjent / ikke godkjent). The three exercise sets are mandatory to be admitted to the final exam. Note that to get an exercise set approved you are required to have solved correctly at least 70% (providing detailed arguments/computations) and made serious and substantial attempts at solving at least 90% of the given exercise set.

You are asked to pay attention to the quality of presentation, in particular, the correctness of mathematical notation and formalism.

Deadline: 21-04-2022, 22h00 (OVSYS)

Problem 1. Süli–Mayers*: Exercises 2.11, 7.6, 10.4, 12.7, 12.8.*

Problem 2. *Construct a quadrature formula of the type:*

$$\int_0^h f(x)dx \approx af(0) + bf(h), \qquad \text{for } h \in (0, \pi)$$

(1) *What are the weights, such that the formula is exact for* $f(x) = \sin x$.
(2) *Show that the formula holds for the interval* $[c, c+h]$.
(3) *Is this formula exact for a constant function,* $f(x) = d$*?*

Problem 3. *Consider the function* $f(x) = ln(x^2 + 1)$ *for* $x \in [-1, 1]$ *and the weighted integral* $I = \int_{-1}^{1} w(x)f(x)dx$, *with* $w(x) = \sqrt{1 - x^2}$.

(1) *Calculate the integral using the trapezium rule and Richardson's extrapolation for* $h_0 = 2$ *and* $h_1 = 1$.
(2) *Calculate the same integral using Gaussian quadrature and three quadrature nodes:*
   - *Construct the orthogonal polynomials to find the three quadrature nodes (Chebyshev polynomials of the second kind),*
   - *calculate the corresponding quadrature weights and*
   - *compare the result to that of (1).*
(3) *(Optional) Compare the computational cost of each method. What happens if you do more steps using Romberg's integration?*

Problem 4. *Recall that Euler's method as well as the implicit Euler method are first order when applied to the autonomous system* $y' = f(y)$. *Show that the following method is of Runge–Kutta type:*

a) use Euler's method with step size $h/2$ *to get from* y_n *to* $y_{n+\frac{1}{2}}$,

b) use implicit Euler's method with step size $h/2$ *to get from* $y_{n+\frac{1}{2}}$ *to* y_{n+1},

find its Butcher tableau and determine its order.

Date: March 17, 2023.

Problem 5. *Consider the implicit Runge–Kutta method and use Newton's method to implement it.*

Problem 6. *Consider the matrix* $A = [a_1|a_2|a_3]$ *with column vectors* $a_1 = (2, 3, 10)^T$, $a_2 = (4, 3, 10)^T$, $a_3 = (-4, 3, 5)^T$. *What is the iteration matrix* T *for the Jacobi method corresponding* A*? Compute* $\rho(T)$ *and determine whether the Jacobi method converges for any choice of initial vector?*

Problem 7. OPTIONAL

(1) *i) Consider the function* $y = (y_1(t), y_2(t))^T : \mathbb{R} \to \mathbb{R}^2$ *and find a system of 1st-order differential equations in autonomous vector form, with initial values equivalent to the following initial value problem for* y: $y_1''(t) - \exp(-y_1'(t)) = y_1(t) - \cos(t)$, $y_2'(t) + ty_1'(t) = (y_2(t))^{\frac{1}{3}}$, $y_1(0) = -2$, $y_1'(0) = 0$, $y_2(0) = 8$.
ii) Compute two steps of Euler's method with step-size $h = 1/2$ *for this system.*

(2) *Consider the nonlinear system* $(xy - 10)y = -10$, $(16x - 80)x + y^2 = -40$. *Compute an approximate solution* $(x^{(1)}, y^{(1)})^T$ *by applying one iteration of Newton's method starting from the initial value* $(0, 0)^T$.

(3) *Find the the LU-decomposition for the matrix* $A = [a_1|a_2|a_3]$ *with column vectors* $a_1 = (3, 6, -3)^T$, $a_2 = (1, 3, 2)^T$, $a_3 = (2, 3, -3)^T$.

Problem 8. *Let* $f(x) = x^5 - 2x^2 + x$, *for* $x \in [0, 2]$. *Calculate the derivative of* $f(x)$ *at* $x = 1$:

(1) *Use forward differences and central differences with* $h = 1$.

(2) *Using* $h_0 = 1$ *and* $h_1 = h_0/2$, *derive a finite difference scheme that is* 4^{th}*-order accurate and use it to calculate the derivative.*

(3) *Find the Lagrange interpolation polynomial with nodes* $x_0 = 0, x_1 = 1, x_2 = 2$ *and calculate the derivative.*

(4) *(Optional) Find the Hermite interpolation polynomial with nodes* $x_0 = 0$ *and* $x_1 = 2$ *and calculate the derivative.*

Were you to implement the schemes from cases (1) and (2) in a computational algorithm, what is the expected result as you decrease the size h*?*

Using the Hermite interpolation polynomial with nodes $x_0 = 0$ *and* $x_1 = 1$ *and* $x_2 = 2$, *what is the value of* $f'(0.5)$*?*

Problem 9. *Consider the boundary value problem*

$$\begin{cases} u_{xx} - u_x = f(x), & \text{for } x \in (0, 1) \\ u(0) = 0, \\ u(1) = \dfrac{2}{\pi^2}, \end{cases}$$

with $f(x) = 2\cos(\pi x)$.

(1) *Construct the discretised linear system of the problem using the finite difference method (central differences) and equidistant grid points on* $[0, 1]$.

(2) *Prove that the constructed scheme has 2-norm stability.*

(3) *Instead of the Dirichlet condition at* $x = 1$, *consider the following Neumann one,* $u_x(1) = -\frac{1}{\pi}$. *Construct a second-order consistent discretised linear system without the use of fictitious nodes.*