MA2501
NUMERICAL METHODS
NTNU, SPRING 2023

Exercise Set 2

Exercise sets are to be handed in individually by each student using the OVSYS system. Each set will be graded (godkjent / ikke godkjent). The three exercise sets are mandatory to be admitted to the final exam. Note that to get an exercise set approved you are required to have solved correctly at least 70% (providing detailed arguments/computations) and made serious and substantial attempts at solving at least 90% of the given exercise set.

You are asked to pay attention to the quality of presentation, in particular, the correctness of mathematical notation and formalism.

Deadline: 10-03-2022, 22h00 (OVSYS)

Problem 1. *Apply Housholder transformations to transform the following matrix A into upper triangular form:*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 2. *Let A be a non-singular matrix of size n and u, v are vectors in $\mathbb{R}^n$. Show that for $v^T A^{-1} u \neq -1$ the following formula holds:*

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1} u}.$$

Problem 3. *Consider a non-singular matrix $A \in \mathbb{R}^{n\times n}$ and its SVD*

$$A = USV^T,$$

where $U = [u_1|\cdots|u_n] \in \mathbb{R}^{n\times n}$ and $V = [v_1|\cdots|v_n] \in \mathbb{R}^{n\times n}$.

i) Show that for any vector $x = \sum_{k=1}^n c_k v_k$ the identity

$$Ax = \sum_{i=1}^n c_i \sigma_i u_i,$$

where σ_i, $1 \leq i \leq n$, are the singular values of the matrix A.

ii) Compute $||A||_2$, $||A^{-1}||_2$, and the condition number $\kappa_2(A)$ (relative to the $||\cdot||_2$-norm) using the singular values of A.

iii) Consider the system $Ax = b$ and the perturbed system $A(x + \Delta x) = b + \Delta b$. Use the matrix V in the SVD of A to find the vectors $b, \Delta b \in \mathbb{R}^n$ such that

$$1)\ ||b||_2 = ||A||_2||x||_2, \quad 2)\ ||\Delta x||_2 = ||A^{-1}||_2||\Delta b||_2, \quad 3)\ \frac{||\Delta x||_2}{||x||_2} = \kappa_2(A)\frac{||\Delta b||_2}{||b||_2}.$$

Date: February 19, 2023.

Problem 4. *Consider a fixed-point method* $x^{(k+1)} = Tx^{(k)} + c$*, where the iteration matrix* $T = T(A)$*, for* A *a symmetric positive definite matrix. Assume that the matrix* $\mathbf{1} - T$ *is similar to a symmetric positive definite matrix. Show that the spectrum of* T *is real and smaller than 1, i.e.,* $-\infty < \sigma(T) < 1$*.*

Problem 5. OPTIONAL

i) Süli–Mayers*: Exercises 5.6, 5.7.*

ii) Recall that with a symmetric positive definite matrix A *comes a scalar product* $(v, u) := \langle u, Av\rangle$*,* $u, v \in \mathbb{R}^n$*. The adjoint of a matrix* $B \in \mathbb{R}^{n\times n}$ *with respect to* $(\cdot,\cdot)$ *is defined to be* $B^* := A^{-1}B^TA$*, i.e.,* $(Bu, v) = (u, B^*v)$ *for all* $u, v \in \mathbb{R}^n$*. If* $B^* = B$ *and* $(Bu, u) > 0$ *for all* $u \neq 0$ *in* $\mathbb{R}^n$*, then* B *is called positive.*

Let $T \in \mathbb{R}^{n\times n}$*. Show that if the matrix* $\mathbf{1} - T^*T$ *is positive with respect to* $(\cdot,\cdot)$*, then* $\rho(T) < 1$*.*

iii) Show that the Gauss–Seidel method converges for any symmetric positive definite matrix A*.*

iv) Consider the fixed-point method $x^{(k+1)} = Tx^{(k)} + c$ *defined in terms of* $T = \mathbf{1} - A$*. Assume* A *to be symmetric positive definite. Find a necessary condition for the convergence of this method.*

Problem 6. *Consider a real-valued function* $f(x)$ *with* $x \in [a, b]$ *that is continuous and* $n + 1$ *times differentiable and* $f^{(n+1)}$ *is continuous in* $[a, b]$*.*

i) (Optional) *Assuming* $n \geq 1$ *equidistant interpolation nodes, i.e.* $x_i = a + ih$*, with* $i = 0, 1, \ldots, n$ *and* h *being the characteristic discretisation size, prove that the error bound of the interpolation polynomial of degree* n *is:*

$$\varepsilon(x) \leq \frac{h^{n+1}}{4(n+1)} M_{n+1}.$$

ii) Interpolate the function $f(x) = \ln(x^2 + 1)$ *with a cubic polynomial in the interpolation nodes* $x_0 = 2$*,* $x_1 = 3$*,* $x_2 = 9$*,* $x_3 = 10$*. Write down the interpolation polynomial and find the error bound at* $x = 6$*.*

iii) For the same function as in ii), using equidistant nodes with $n = 2$*,* $n = 3$ *and* $n = 10$*, calculate the error bound of the interpolation polynomial for* $x \in [2, 10]$*. What is your conclusion?*

iv) (Optional) *Using Python, calculate the maximum error and compare with the error bound. How much do they differ?*

Problem 7. *Construct the minimax polynomial* $p_1 \in \mathcal{P}_\infty$ *on the interval* $[-1, 4]$ *for the function* $f(x) = x^2$*.*

Problem 8. *If* x_k*,* $k = 0, 1, \ldots, n$ *are quadrature points, write the Newton–Cotes formula and prove that it is exact for all polynomials of degree* $n + 1$ *if* n *is even.*

Hint: Prove symmetry of the integration weights $w_k = w_{n-k}$ *and consider that all polynomials of degree* $n + 1$ *can be written in the form* $q_{n+1} = \alpha\left(x + \frac{a+b}{2}\right)^{n+1} + p_n$*.*

Problem 9. *Consider the function* $f(x) = \sin(x) + 1$ *for* $x \in [0, \alpha 2\pi]$*, where* $\alpha \in \mathbb{N}$ *and* $\alpha \geq 1$*. Consider a decomposition of* $[0, \alpha 2\pi]$ *in* m *subdomains with* $h = \frac{\alpha 2\pi}{m}$ *and calculate the integral using the trapezoidal rule. Prove that the integration error is 0.*

Similarly, prove that the integration by means of the composite trapezoidal rule of smooth periodic functions defined in $[-\infty, \infty]$ *and* $2k$ *times differentiable is* $\mathcal{O}(h^{2k})$ *accurate, with* $k \geq 1$*.*

Problem 10. OPTIONAL

Süli–Mayers*: Exercises 6.10, 6.11, 7.7, 7.11, 8.8.*