MA2501
NUMERICAL METHODS
NTNU, SPRING 2023

EXERCISE SET 1

Exercise sets are to be handed in individually by each student using the OVSYS system. Each set will be graded (godkjent / ikke godkjent). The three exercise sets are mandatory to be admitted to the final exam. Note that to get an exercise set approved you are required to have solved correctly at least 70% (providing detailed arguments/computations) and made serious and substantial attempts at solving at least 90% of the given exercise set.

You are asked to pay attention to the quality of presentation, in particular, the correctness of mathematical notation and formalism.

Deadline: 10-02-2022, 22h00 (OVSYS)

Problem 1. *(Taylor's theorem)*

i) Consider the function $f(x) = \ln(\cos(x))$ *and compute its Taylor polynomial of degree four around the point zero.*

ii) Consider the function $f(x) = \sin(x)\ln(1+x)$ *and compute its Taylor polynomial of degree four around the point zero.*

iii) Let $n \geq 0$ *be a non-negative integer. Suppose that* f *is a continuous real-valued function on the* $[a,b] \subset \mathbb{R}$ *with derivatives* $f^{(n)}$ *of order up to and including* n *defined and continuous on* $[a,b]$*. Moreover,* $f^{(n)}$ *is differentiable on the open interval* $]a,b[$*, and* $f^{(n+1)}$ *is integrable on* $]a,b[$*. Show that for each* x *in* $[a,b]$ *the remainder term in the Taylor expansion of* f *has the form*

$$\int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt.$$

Problem 2. *(Floating point numbers)*

For details on floating point numbers, please consult E. Celledoni's note "Numerical Methods: brief introduction to floating point numbers" (link)

i) Let $x = 0.d_1 \cdots d_k \cdots \times 10^n$ *in decimal representation (basis* $b = 10$*). Aiming at a* k*-digit floating point representation, we consider chopping instead of rounding, i.e., we keep the* k *first digits and throw away the rest*

$$f\ell(x) = 0.d_1 \cdots d_k \times 10^n$$

i.a) Show that 10^{-k-1} *is a bound for the relative error when using* k*-digit chopping.*

i.b) Show that $0.5 \cdot 10^{-k-1}$ *is a bound for the relative error when using* k*-digit rounding.*

ii) Determine the five-digit (a) chopping and (b) rounding values of the irrational number π*.*

Date: January 20, 2023.

Problem 3. *We consider Newton's formula for computing the square root* $\alpha = \sqrt{a}$, $a > 0$.

i) Show that the iteration has the following form

$$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{a}{x_n}\Big).$$

ii) Show that

$$\frac{x_{n+1} - \alpha}{(x_n - \alpha)^2} = \frac{1}{2x_n}$$

and obtain the asymptotic error constant.

iii) Find analogous formulas for the cubic root.

Problem 4. *(Norms) i) For a normed vector space, show the inverted triangle equation*

$$||x - y|| \geq \big|||x|| - ||y||\big|.$$

ii) Let A *be a* $n \times n$*-matrix over the real numbers. Show that for the vector norms* $||\cdot||_\infty$ *and* $||\cdot||_1$ *the induced matrix norms are*

$$||A||_\infty = \max_{j=1,\dots,n} \sum_{k=1}^{n} |a_{jk}|$$

respectively

$$||A||_1 = \max_{k=1,\dots,n} \sum_{j=1}^{n} |a_{jk}|.$$

iii) Show that for any matrix $A \in \mathbb{R}^{n\times n}$

$$||A||_F := \Big(\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2\Big)^{1/2}$$

defines a matrix norm, and that

$$||A||_2 \leq ||A||_F \leq n^{1/2}||A||_2.$$

Problem 5. *Compute the spectral radius of the matrices*

$$T_1 = \begin{pmatrix} 0 & 1/2 & -1/2 \\ -1 & 0 & -1 \\ 1/2 & 1/2 & 0 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 0 & 1/2 & -1/2 \\ 0 & -1/2 & -1/2 \\ 0 & 0 & -1/2 \end{pmatrix}.$$

Problem 6. Süli–Mayers*: Ex. 1.6, Ex. 2.8, Ex. 2.9, Ex. 2.10, Ex. 4.7, Ex. 4.8.*