MA2501
NUMERICAL METHODS
NTNU, SPRING 2022

Exercise Set 3

Exercise sets are to be handed in individually by each student using the OVSYS system[1]. Each set will be graded (godkjent / ikke godkjent). The three exercise sets are mandatory to be admitted to the final exam. Note that to get an exercise set approved you are required to have solved correctly at least 60% (providing detailed arguments/computations) and made serious and substantial attempts at solving at least 90% of a given exercise set. You are allowed to use a computer only for Problem 3, and submit your code file with description of your findings. For other parts, submit a report in the form of a PDF file.

You are asked to pay attention to the quality of presentation, in particular, the correctness of mathematical notation and formalism.

Deadline: 28-03-2022, 16h00 (OVSYS)

Problem 1. *The following way of constructing orthogonal polynomials is called* ***Gram–Schmidt orthogonalisation:***
For a given weight $w(x)$ on (a,b) and the zero-th polynomial $\varphi_0 \in P_0$, we successively construct polynomials $\varphi_j \in P_j$ for $j = 1, 2...$ such that

$$\langle \varphi_j, \varphi_k \rangle = \begin{cases} 0, & \text{if } j \neq k, \\ 1, & \text{if } j = k, \end{cases}$$

where the inner product is given by

$$\langle \varphi_j, \varphi_k \rangle := \int_a^b \varphi_j(x)\varphi_k(x)w(x)\,\mathrm{d}x.$$

For instance, φ_1 is determined by choosing c_0 such that

$$\tilde{\varphi}_1(x) = x - c_0\varphi_0(x), \quad \langle \tilde{\varphi}_1, \varphi_0 \rangle = \langle x, \varphi_0 \rangle - c_0 \langle \varphi_0, \varphi_0 \rangle = 0.$$

After finding such $\tilde{\varphi}_1$, we normalize $\varphi_1 = \tilde{\varphi}_1 / \|\tilde{\varphi}_1\|_2$ so that $\langle \varphi_1, \varphi_1 \rangle = 1$ is satisfied.
For φ_2, we choose d_0 and d_1 for $\tilde{\varphi}_2$ of the form

$$\tilde{\varphi}_2(x) = x^2 - (d_1\varphi_1(x) + d_0\varphi_0(x))$$

satisfying $\langle \tilde{\varphi}_2, \varphi_1 \rangle = 0$ and $\langle \tilde{\varphi}_2, \varphi_0 \rangle = 0$. This means

$$\begin{cases} \langle x^2, \varphi_1 \rangle - d_1 \langle \varphi_1, \varphi_1 \rangle - d_0 \langle \varphi_0, \varphi_1 \rangle = 0 \\ \langle x^2, \varphi_0 \rangle - d_1 \langle \varphi_1, \varphi_0 \rangle - d_0 \langle \varphi_0, \varphi_0 \rangle = 0. \end{cases}$$

After finding such $\tilde{\varphi}_2$, we again normalize $\varphi_2 = \tilde{\varphi}_2 / \|\tilde{\varphi}_2\|_2$. The same procedure generates $\varphi_3, \varphi_4, \ldots$.

Date: March 7, 2022.

[1] A link can be found here: `https://wiki.math.ntnu.no/ma2501/2022v/start`

(1) *For* $[a,b] = [-1,1]$, $w(x) = 1$, *and* $\varphi_0 = 1/\sqrt{2}$, *construct* φ_1, φ_2, *and* φ_3 *by the above Gram–Schmidt orthogonalisation.*

(2) *The same procedure works for unbounded domain, e.g.,* $\mathbb{R} = (-\infty, \infty)$. *Consider* $w(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ *for* $x \in \mathbb{R}$. *Given* $\varphi_0 = 1$, *construct* φ_1, φ_2, *and* φ_3. *You can use the following without proof*

$$\int_{-\infty}^{\infty} w(x) x^k \,\mathrm{d}x = \begin{cases} 0, & \text{if } k \text{ is odd} \\ k!! = k(k-2)\cdots, & \text{if } k \text{ is even} \end{cases}$$

where $k!!$ *is the double factorial of* k.

Problem 2. *For a weight function* $w(x) = \frac{1}{\sqrt{1-x^2}}$ *on* $x \in (-1,1)$, *and for given any small* $\varepsilon > 0$ *and any large* $M > 0$, *constrct such a function* $f \in C[-1,1]$ *satisfying the following inequalities:*

$$\|f\|_2 < \varepsilon, \quad \|f\|_\infty > M,$$

where

$$\|f\|_2 := \left(\int_{-1}^{1} w(x) |f(x)|^2 \,\mathrm{d}x \right)^{1/2}.$$

Note that f *can depend on* ε *and* M. *(Proof of Lemma 8.1 (ii) with a specific* $w(x)$*.)*

Problem 3. *Using Jupyter notebook, compute the quadrature error for the following integrals*

$$I(f_1) = \int_{-1}^{1} |x|^3 \,\mathrm{d}x, \quad I(f_2) = \int_{-1}^{1} \frac{1}{1+25x^2} \,\mathrm{d}x$$

with (i) the composite trapezoidal rule, and (ii) the composite Simpson's rule. Make log-log plots of quadrature errors for both methods for f_1 *and* f_2, *where the x-axis represents the number of quadrature nodes* n *and y-axis represents the absolute error* $|I(f) - Q_n(f)|$. *You can use the exact integral value* $I(f_2) = \frac{2}{5} tan^{-1}(5)$ *for calculating the error. What convergence rate do you observe for each case? Note that if you observe the second order convergence* $|I(f) - Q_n(f)| = \mathcal{O}(n^{-2})$, *the error plot should look like the below:*

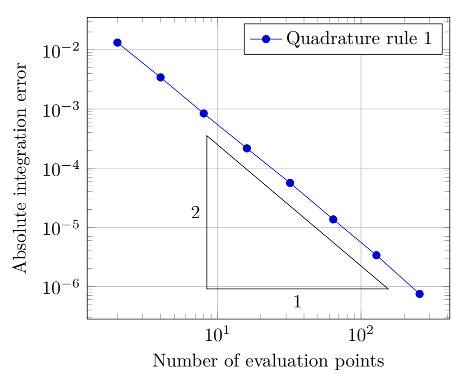

FIGURE 1. Example of the error plot

Try to give theoretical explanations for the error convergence rate you are observing.

Problem 4. Süli–Mayers*: Ex. 6.5, Ex. 6.6, Ex. 7.7, Ex. 7.13, Ex. 8.9, Ex. 12.1, Ex. 12.3,*