MA2501
NUMERICAL METHODS
NTNU, SPRING 2020

Exercise Set 2

Exercise sets are to be handed in individually by each student using the OVSYS system[1]. Each set will be graded (godkjent / ikke godkjent). The three exercise sets are mandatory to be admitted to the final exam. Note that to get an exercise set approved you are required to have solved correctly at least 70% (providing detailed arguments/computations) and made serious and substantial attempts at solving at least 90% of a given exercise set.

You are asked to pay attention to the quality of presentation, in particular, the correctness of mathematical notation and formalism.

Deadline: 28-02-2022, 16h00 (OVSYS)

Problem 1. (1) *Show that the function* $f(x) = (x+1)(x-1)/3$ *has a unique fixed point in the interval* $[-1, 1]$*. What can you say about the interval* $[3, 4]$.

(2) *Compute the spectral radius of the matrices*

$$T_1 = \begin{pmatrix} 0 & 1/2 & -1/2 \\ -1 & 0 & -1 \\ 1/2 & 1/2 & 0 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 0 & 1/2 & -1/2 \\ 0 & -1/2 & -1/2 \\ 0 & 0 & -1/2 \end{pmatrix}.$$

(3) *Show that for any matrix* $A \in \mathbb{R}^{n\times n}$

$$||A||_F := \Big(\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2\Big)^{1/2}$$

defines a matrix norm. (This is the so-called Frobenius norm.) Use the Cauchy–Schwarz inequality (Süli-Mayers, Lemma 2.2, page 59) to show that for any matrix $A \in \mathbb{R}^{n\times n}$ *and any vector* $x \in \mathbb{R}^n$

$$||Ax||_2 \leq ||A||_F ||x||_2.$$

Problem 2. *Consider the system*

$$\begin{aligned} 10x_1 - x_2 + 2x_3 &= 6 \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\ 2x_1 - x_2 - 10x_3 - x_4 &= -11 \\ 3x_2 - x_3 + x_4 &= 15. \end{aligned}$$

Find its exact solution. Write down the Jacobi iterative method and generate the first 3 entries, $x^{(1)}, x^{(2)}, x^{(3)}$ *in the sequence of approximations* $\{x^{(n)}\}_{n>0}$, $x^{(0)} = (0,0,0,0)^T$*. Repeat the approximation using the Gauss–Seidel iterative method.*

Date: February 7, 2022.

[1] A link can be found here: `https://wiki.math.ntnu.no/ma2501/2022v/start`

Optional [You do not need to submit this part and you can use a computer for this specific problem]: Find for both methods the n such

$$\frac{||x^{(n)} - x^{(n-1)}||_\infty}{||x^{(n)}||_\infty} < 10^{-3}.$$

Problem 3. *Looking at the system*

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1 \\ 2x_1 + 2x_2 + 2x_3 &= 4 \\ -x_1 - x_2 + 2x_3 &= -5, \end{aligned}$$

and using the results from Problem 1(2), what can you say about applying the Jacobi iterative method and the Gauss–Seidel iterative method, both for initial value $x^{(0)} = (0, 0, 0)$?

Optional [You do not need to submit this part and you can use a computer for this specific problem]: How good is the approximation $x^{(23)}$ using the Gauss–Seidel iterative method and the $||\cdot||_\infty$-norm?

Problem 4. *Use LU factorisation to find the solution of the system (provide all details):*

$$\begin{aligned} x_1 + x_2 + 3x_4 &= 8 \\ 2x_1 + x_2 - x_3 + x_4 &= 7 \\ 3x_1 - x_2 - x_3 + 2x_4 &= 14 \\ -x_1 + 2x_2 + 3x_3 - x_4 &= -7 \end{aligned}$$

Problem 5. *Formulate the problem of finding a straight line $y = x_1 + tx_2$ fitting the following points in the (t, y)-plane*

$$\begin{gathered} (1, 1.4501) \\ (2, 1.7311) \\ (3, 3.1068) \\ (4, 3.9860) \\ (5, 5.3913) \end{gathered}$$

as a least squares problem and solve it, i.e., find x_1, x_2.

Problem 6. Süli–Mayers*: Ex. 1.10, Ex. 2.7, Ex. 2.14, Ex. 2.15, Ex. 5.1, Ex. 5.2.*