NTNU
Norwegian University of Science and Technology

Department of Mathematical Sciences

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Date Signature

Problem 1 (15 points)

(i) (8 point) Let M be a symmetric positive definite $n \times n$-matrix with eigenvalues $0 < \lambda_n \leq \cdots \leq \lambda_1$. Define $||x||_M^2 := x^T M x$, $x \in \mathbb{R}^n$. Show that $||x||_M \leq ||y||_M$ implies that $||x||_2 \leq \sqrt{\lambda_1 \lambda_n^{-1}} ||y||_2$.

(ii) (7 point) Find a counterexample to the claim that the norm $||M||_{\max} := \max_{i,j} |m_{ij}|$, for matrices $M \in \mathbb{R}^{n \times n}$, satisfies $||M_1 M_2||_{\max} \leq ||M_1||_{\max} ||M_2||_{\max}$.

[Solution] (i) M is a symmetric positive definite matrix with $M u_j = \lambda_j u_j$ and $u_i^T u_j = \delta_{ij}$, $i, j = 1, \ldots, n$. For $x = \sum_{j=1}^n a_j u_j$ and $y = \sum_{j=1}^n b_j u_j$, we have $Ax = \sum_{j=1}^n a_j \lambda_j u_j$ and $Ay = \sum_{j=1}^n b_j \lambda_j u_j$. Therefore, $||x||_M^2 = \sum_{j=1}^n a_j^2 \lambda_j$. The 2-norms are $||x||_2^2 = \sum_{j=1}^n a_j^2$ and $||y||_2^2 = \sum_{j=1}^n b_j^2$. Now, we have that $\lambda_n ||x||_2^2 \leq ||x||_M^2$ and $||y||_M^2 \leq \lambda_1 ||y||_2^2$. This then implies

$$||x||_2^2 \leq \frac{1}{\lambda_n} ||x||_M^2 \leq \frac{1}{\lambda_n} ||y||_M^2 \leq \frac{\lambda_1}{\lambda_n} ||y||_2^2,$$

which implies the result.

(ii) Consider the 2×2-matrix M with entries $m_{ij} = 1$ for $i, j = 1, 2$.

Problem 2 (15 points)

(i) (8 point) Let $M \in \mathbb{R}^{n \times n}$ and assume that $|| \cdot ||$ is a matrix norm. Suppose that $I - M$ is non-singular. Show that $||(I - M)^{-1} - I|| \leq ||M|| \, ||(I - M)^{-1}||$. Here I denotes the identity matrix in $\mathbb{R}^{n \times n}$.

(ii) (7 point) Let $M \in \mathbb{R}^{n \times n}$. Assume that $(I - M)^{-1}$ exists and that $|| \cdot ||$ is a matrix norm. Show that $(1 + ||M||)^{-1} \leq ||(I - M)^{-1}||$. Here I denotes the identity matrix in $\mathbb{R}^{n \times n}$.

Solution:

(i) The matrix $I-M$ being non-singular implies that $M=(I-M)[(I-M)^{-1}-I]$. Therefore, we have $(I-M)^{-1}M=(I-M)^{-1}-I$ and $||(I-M)^{-1}-I|| \leq ||(I-M)^{-1}||\ ||M||$, which is the inequality we where looking for.

(ii) As $(I-M)^{-1}$ exists, we have $||M|| = ||(I-M)[(I-M)^{-1}-I]|| \leq ||(I-M)||\ ||(I-M)^{-1}-I||$. Therefore, we have $(1+||M||)^{-1} \leq ||I-M||^{-1} = ||M||(||M||\ ||I-M||)^{-1}$. Now, from (i) we have

$$||M||\ (||M||\ ||I-M||)^{-1} \leq \frac{||(I-M)||\ ||(I-M)^{-1}-I||}{||M||\ ||I-M||} \leq ||(I-M)^{-1}||.$$

Problem 3 (15 points) The **Chebyshev–Gauss quadrature** of Type 2 is one of Gaussian quadrature rules approximating

$$I(f) := \int_{-1}^{1} f(x)\sqrt{1-x^2}\,\mathrm{d}x$$

by

$$Q_n(f) := \sum_{i=0}^{n} W_i f(x_i), \quad n \geq 0,$$

where the quadrature nodes x_i and quadrature weights W_i are explicitly given by

$$x_i = \cos\left(\frac{(i+1)\pi}{(n+2)}\right), \quad W_i = \frac{\pi}{n+2}\sin^2\left(\frac{(i+1)\pi}{(n+2)}\right).$$

This Gaussian rule is constructed for the weight $\sqrt{1-x^2}$ in the integral.

(i) (5 points) What is the smallest number n for integrating $f(x)=x^3+x^2$ exactly by this rule? In other words, find the smallest n satisfying $Q_n(f)=I(f)$ for this function. Explain why.

(ii) (5 points) Calculate $I(f)$ by calculating $Q_n(f)$ for $f(x)=x^3+x^2$.

(iii) (5 points) Prove that $Q_n(f)$ is exact for any integrable odd function f with any $n \geq 0$.

[Solution] (i) Since Gaussian quadrature Q_n is exact for all polynomials of degree at most $2n+1$, and $f(x)$ is a third degree polynomial, we only need $n = 1$.

(ii) Due to the reasoning above,

$$I(f) = Q_1(f) = \sum_{i=0}^{1} \frac{\pi}{3} \sin^2\left(\frac{(i+1)\pi}{3}\right) f\left(\cos\left(\frac{(i+1)\pi}{3}\right)\right)$$
$$= \frac{\pi}{3}\frac{3}{4}\left(f\left(\frac{-1}{2}\right) + f\left(\frac{1}{2}\right)\right) = \frac{\pi}{8}.$$

(iii) For any integrable odd function, we have

$$I(f) = \int_{-1}^{1} f(x)\sqrt{1-x^2}\,\mathrm{d}x = 0,$$

due to the odd property of $f(x) = -f(-x)$. When n is even, then we have odd number of nodes, and we see that $x_i = -x_{n-i}$ and $W_i = W_{n+1-i}$ for $i = 0, \ldots, \frac{n}{2}-1$, and the middle point is always $x_{n/2} = 0$. Thus,

$$Q_n(f) = \sum_{i=0}^{n} W_i f(x_i) = \sum_{i=0}^{\frac{n}{2}-1} W_i(f(x_i) - f(x_i)) + W_{n/2} f(0) = 0.$$

When n is odd, then we have even number of nodes, and we see that $x_i = -x_{n+1-i}$ and $W_i = W_{n+1-i}$ for $i = 0, \ldots, \frac{n+1}{2}$. Thus,

$$Q_n(f) = \sum_{i=0}^{n} W_i f(x_i) = \sum_{i=0}^{\frac{n+1}{2}} W_i(f(x_i) - f(x_i)) = 0.$$

Thus the claim is proved.

Problem 4 (15 points) The **Chebyshev–Gauss quadrature** is one of Gaussian quadrature rules approximating

$$I(f) := \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,\mathrm{d}x$$

by

$$Q_n(f) := \sum_{i=0}^{n} W_i f(x_i), \quad n \geq 0,$$

where the quadrature nodes x_i and quadrature weights W_i are explicitly given by

$$x_i = \cos\left(\frac{2i+1}{2(n+1)}\pi\right), \quad W_i = \frac{\pi}{n+1}.$$

This Gaussian rule is constructed for the weight function $\frac{1}{\sqrt{1-x^2}}$ in the integral.

(i) (5 points) What is the smallest number n for integrating $f(x) = x^4$ exactly by this rule? In other words, find the smallest n satisfying $Q_n(f) = I(f)$ for this function. Explain why.

(ii) (5 points) Calculate $I(f)$ by calculating $Q_n(f)$ for $f(x) = x^4$.

(iii) (5 points) Prove that $Q_n(f)$ is exact for any integrable odd function f with any $n \geq 0$.

[Solution] (i) Since Gaussian quadrature Q_n is exact for all polynomials of degree at most $2n+1$, and $f(x)$ is the fourth degree polynomial, we only need $n = 2$.

(ii) Using the above reasoning, we have

$$\begin{aligned} I(f) = Q_2(f) &= \sum_{i=0}^{2} \frac{\pi}{3} f\left(\cos\left(\frac{(2i+1)\pi}{6}\right)\right) \\ &= \frac{\pi}{3}\left(\left(-\frac{\sqrt{3}}{2}\right)^4 + 0 + \left(\frac{\sqrt{3}}{2}\right)^4\right) = \frac{3\pi}{8}. \end{aligned}$$

(iii) For any integrable odd function, we have

$$I(f) = \int_{-1}^{1} f(x) \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x = 0,$$

due to the odd property of $f(x) = -f(-x)$. When n is even, then we have odd number of nodes, and we see that $x_i = -x_{n-i}$ and $W_i = W_{n+1-i}$ for $i = 0, \ldots, \frac{n}{2}-1$, and the middle point is always $x_{n/2} = 0$. Thus,

$$Q_n(f) = \sum_{i=0}^{n} W_i f(x_i) = \sum_{i=0}^{\frac{n}{2}-1} W_i(f(x_i) - f(x_i)) + W_{n/2} f(0) = 0.$$

When n is odd, then we have even number of nodes, and we see that $x_i = -x_{n+1-i}$ and $W_i = W_{n+1-i}$ for $i = 0, ..., \frac{n+1}{2}$. Thus,

$$Q_n(f) = \sum_{i=0}^{n} W_i f(x_i) = \sum_{i=0}^{\frac{n+1}{2}} W_i(f(x_i) - f(x_i)) = 0.$$

Thus the claim is proved.

Problem 5 (15 points)

Let α be a root of $f(x) = 0$ of multiplicity two and assume that the function f is sufficiently smooth close to α. Show that if the method

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}$$

converges to α, it does so at least quadratically. Determine the condition for which the order of convergence is exactly two.

Solution:

Consider Newton's method $x_{n+1} = \phi(x_n)$ with

$$\phi(x_n) = x - 2\frac{f(x_n)}{f'(x_n)}.$$

As α is a zero of multiplicity two, we have

$$f(x) = (x - \alpha)^2 g(x),$$

where $g(\alpha)$ is different from zero and

$$g(x) = \frac{1}{2!}f''(\alpha) + \frac{x - \alpha}{3!}f'''(\alpha) + \cdots .$$

Therefore

$$\phi(x) = x - \frac{2(x - \alpha)^2 g(x)}{2(x - \alpha)g(x) + (x - \alpha)^2 g'(x)} = x - (x - \alpha)\frac{g(x)}{g(x) + \frac{1}{2}(x - \alpha)g'(x)}.$$

Then

$$\phi'(x) = 1 - \frac{g(x)}{g(x) + \frac{1}{2}(x-\alpha)g'(x)} - (x-\alpha)\Big(\frac{g(x)}{g(x) + \frac{1}{2}(x-\alpha)g'(x)}\Big)'$$

and $\phi'(\alpha) = 0$, which implies at least quadratic convergence. Second derivation of ϕ at α gives

$$\phi''(\alpha) = \frac{g'(\alpha)}{g(\alpha)} = \frac{1}{3}\frac{f'''(\alpha)}{f''(\alpha)}.$$

If $f'''(\alpha)$ is different from zero, then we have convergence of order two exactly.

Problem 6 (15 points)

Let α be a root of $f(x) = 0$ of multiplicity three and assume that the function f is sufficiently smooth close to α. Show that if the method

$$x_{n+1} = x_n - 3\frac{f(x_n)}{f'(x_n)}$$

converges to α, it does so at least quadratically. Determine the condition for which the order of convergence is exactly two.

Solution:

Completely analog to above argument, just replace 2 by 3 in ϕ. In fact, the proof works for general m, that is, the method $x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$ and α a root of $f(x) = 0$ of multiplicity m.

Problem 7 (20 points) Consider the following **Lagrange interpolation problem**: construct a polynomial p_2 of degree at most 2 interpolating the function $f(x) = x^m$ at $x_0 = 0$, $x_1 = 2$, and $x_2 = 4$, where m is an integer with $m \geq 4$.

(i) (5 points) Solve the problem and obtain the solution p_2.

(ii) (7 points) Obtain the absolute upper bound $C(m)$ depending on m such that

$$|f(x) - p_2(x)| \leq C(m),$$

for any $x \in [0, 4]$.

(iii) (8 points) Let $g(x) := f(x) - p_2(x)$. Prove that there exists at least one solution for $g''(x) = 0$ where $x \in (0, 4)$.

[Solution] (i) Using the Lagrange basis functions, we have

$$\begin{aligned} p_2(x) &= \frac{(x-2)(x-4)}{(0-2)(0-4)} f(0) + \frac{x(x-4)}{(2-0)(2-4)} f(2) + \frac{x(x-2)}{(4-0)(4-2)} f(4) \\ &= (2^{2m-3} - 2^{m-2})x^2 + (2^m - 2^{2m-2})x. \end{aligned}$$

(ii) Since $f(x)$ is infinitely smooth and we can apply Theorem 6.2 in Süli–Mayers and obtain: for a given $x \in [0, 4]$ there exists $\xi \in (0, 4)$ such that

$$f(x) - p_2(x) = \frac{f^{(3)}(\xi)}{3!} x(x-2)(x-4).$$

Here, note that $f^{(3)}(\xi) = m(m-1)(m-2)\xi^{m-3} \leq m(m-1)(m-2)4^{m-3}$ and $|x(x-2)(x-4)|$ attains its maxima $\frac{16\sqrt{3}}{9}$ at $x = 2 \pm \frac{2\sqrt{3}}{3}$, thus,

$$|f(x) - p_2(x)| = \left| \frac{f^{(3)}(\xi)}{3!} x(x-2)(x-4) \right| \leq \frac{8\sqrt{3}}{27} m(m-1)(m-2)4^{m-3}.$$

Thus the upper bound is $C(m) = \frac{8\sqrt{3}}{27} m(m-1)(m-2)4^{m-3}$.

(iii) We have three distinct zeros for the function $g(x) := f(x) - p_2(x)$ at $x = 0, 2, 4$. We also know that g is infinitely smooth, thus we can use the Rolle's theorem: there exist two distinct numbers $\xi \in (0, 4)$ such that $g'(\xi) = 0$. Using the Rolle's theorem again, there exists at least one $\xi^* \in (0, 4)$ such that $g''(\xi^*) = 0$.

Problem 8 (20 points) Consider the following **Lagrange interpolation problem**: construct a polynomial p_2 of degree at most 2 interpolating the function $f(x) = e^{ax} - 1$ at $x_0 = 0$, $x_1 = 2$, and $x_2 = 4$, where $a > 0$.

(i) (5 points) Solve the problem and obtain the solution p_2.

(ii) (7 points) Obtain the absolute upper bound $C(a)$ depending on a such that

$$|f(x) - p_2(x)| \leq C(a),$$

for any $x \in [0, 4]$.

(iii) (8 points) Let $g(x) := f(x) - p_2(x)$. Prove that there exists at least one solution for $g''(x) = 0$ where $x \in (0, 4)$.

[Solution]

(i) Using the Lagrange basis functions, we have

$$\begin{aligned} p_2(x) &= \frac{(x-2)(x-4)}{(0-2)(0-4)} f(0) + \frac{x(x-4)}{(2-0)(2-4)} f(2) + \frac{x(x-2)}{(4-0)(4-2)} f(4) \\ &= \frac{1}{8}(e^{4a} - 2e^{2a} + 1)x^2 + (e^{2a} - \frac{e^{4a}}{4} - \frac{3}{4})x. \end{aligned}$$

(ii) Since $f(x)$ is infinitely smooth and we can apply Theorem 6.2 in Süli–Mayers and obtain: for a given $x \in [0, 4]$ there exists $\xi \in (0, 4)$ such that

$$f(x) - p_2(x) = \frac{f^{(3)}(\xi)}{3!} x(x-2)(x-4).$$

Here, note that $f^{(3)}(\xi) = a^3 e^{a\xi} \leq a^3 e^{4a}$ and $|x(x-2)(x-4)|$ attains its maxima $\frac{16\sqrt{3}}{9}$ at $x = 2 \pm \frac{2\sqrt{3}}{3}$, thus,

$$|f(x) - p_2(x)| = \left| \frac{f^{(3)}(\xi)}{3!} x(x-2)(x-4) \right| \leq \frac{8\sqrt{3}}{27} a^3 e^{4a}.$$

Thus the upper bound is $C(a) = \frac{8\sqrt{3}}{27} a^3 e^{4a}$.

(iii) We have three distinct zeros for the function $g(x) := f(x) - p_2(x)$ at $x = 0, 2, 4$. We also know that g is infinitely smooth, thus we can use the Rolle's theorem: there exist two distinct numbers $\xi \in (0, 4)$ such that $g'(\xi) = 0$. Using the Rolle's theorem again, there exists at least one $\xi^* \in (0, 4)$ such that $g''(\xi^*) = 0$.

Problem 9 (20 points)

(20 points) Consider the C^1-function $f(t,y) = (f_1(t,y),\ldots,f_n(t,y))^T : [u,v]\times\mathbb{R}^n \to \mathbb{R}^n$. Let $A_{ij} \in \mathbb{R}_+$, $i,j = 1,\ldots,n$, be the constant entries of the $n\times n$-matrix A. Suppose that on $[u,v]\times\mathbb{R}^n$ we have

$$\left|\frac{\partial f_i(t,y)}{\partial y_j}\right| \le A_{ij}, \qquad i,j = 1,\ldots,n.$$

Determine a Lipschitz constant L of $f(t,y)$ in the 1-norm $||\cdot||_1$. Give L in terms of the corresponding matrix norm of A. (Hint: use the mean value theorem for functions of several variables.)

Solution:

Mean value theorem for functions of several variables:

$$f_i(t,y) - f_i(t,\hat{y}) = \sum_{j=1}^{n} \frac{\partial f_i(t,y^*(i))}{\partial y_j}(y_j - \hat{y}_j).$$

The point $y^*(i)$ lies on the line connecting y and $\hat{y}$ in $\mathbb{R}^n$. Note that it depends on i in the sense that it may differ for different components $f_i = f_i(t,y)$. We compute now with the 1-norm $||\cdot||_1$:

$$\begin{aligned}
||f(t,y) - f(t,\hat{y})||_1 &= \sum_{i=1}^{n} \Big| f_i(t,y) - f_i(t,\hat{y}) \Big| \\
&\le \sum_{i=1}^{n} \Big| \sum_{j=1}^{n} \frac{\partial f_i(t,y^*(i))}{\partial y_j}(y_j - \hat{y}_j) \Big| \\
&\le \sum_{i=1}^{n} \Big| \sum_{j=1}^{n} \frac{\partial f_i(t,y^*(i))}{\partial y_j} \Big| \, \Big| y_j - \hat{y}_j \Big| \\
&\le \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \, \Big| y_j - \hat{y}_j \Big| \\
&= \sum_{i=1}^{n} A_{ij} \sum_{j=1}^{n} \, \Big| y_j - \hat{y}_j \Big| \\
&\le \Big(\max_k \sum_{i=1}^{n} A_{ik} \Big) \sum_{j=1}^{n} \, \Big| y_j - \hat{y}_j \Big|.
\end{aligned}$$

Hence, defining $L := \max_j \sum_{i=1}^{n} A_{ij} = ||A||_1$, we find

$$||f(t,y) - f(t,\hat{y})||_1 \le L||y - \hat{y}||_1.$$

Problem 10 (20 points)

(20 points) Consider the C^1-function $f(t,y) = (f_1(t,y),\dots,f_n(t,y))^T : [u,v]\times\mathbb{R}^n \to \mathbb{R}^n$. Let $A_{ij}\in\mathbb{R}_+$, $i,j=1,\dots,n$, be the constant entries of the $n\times n$-matrix A. Suppose that on $[u,v]\times\mathbb{R}^n$ we have

$$\left|\frac{\partial f_i(t,y)}{\partial y_j}\right| \le A_{ij}, \qquad i,j=1,\dots,n.$$

Determine a Lipschitz constant L of $f(t,y)$ in the ∞-norm $||\cdot||_\infty$. Give L in terms of the corresponding matrix norm of A. (Hint: use the mean value theorem for functions of several variables.)

Solution:

Mean value theorem for functions of several variables:

$$f_i(t,y) - f_i(t,\hat{y}) = \sum_{j=1}^{n} \frac{\partial f_i(t,y^*(i))}{\partial y_j}(y_j - \hat{y}_j).$$

The point $y^*(i)$ lies on the line connecting y and $\hat{y}$ in $\mathbb{R}^n$. Note that it depends on i in the sense that it may differ for different components $f_i = f_i(t,y)$. We compute now with the ∞-norm $||\cdot||_\infty$.

$$\begin{aligned}
||f(t,y) - f(t,\hat{y})||_\infty &= \max_i \Big|f_i(t,y) - f_i(t,\hat{y})\Big| \\
&\le \max_i \Big|\sum_{j=1}^{n} \frac{\partial f_i(t,y^*(i))}{\partial y_j}(y_j - \hat{y}_j)\Big| \\
&\le \max_i \Big|\sum_{j=1}^{n} \frac{\partial f_i(t,y^*(i))}{\partial y_j}\Big| \, \Big|y_j - \hat{y}_j\Big| \\
&\le \max_i \sum_{j=1}^{n} A_{ij} \,\Big|y_j - \hat{y}_j\Big| \\
&\le \left(\max_k \sum_{i=1}^{n} A_{ki}\right) \max_j \Big|y_j - \hat{y}_j\Big|.
\end{aligned}$$

Hence, defining $L := \max_j \sum_{i=1}^{n} A_{ji} = ||A||_\infty$, we find

$$||f(t,y) - f(t,\hat{y})||_\infty \le L||y-\hat{y}||_\infty.$$

Problem 11 (15 points) Consider the following **boundary value problem** for the unknown function $u(x)$:

$$u_{xx} + 4u_x = f(x), \quad 0 < x < 1, \quad u(0) = 1, \quad u(1) = 4,$$

where $f(x) = e^{(x-1/2)}$.

(i) (8 points) Construct a finite difference method using central differences to approximate both u_{xx} and u_x with equidistant grid points on $[0, 1]$. In other words, obtain the discretized linear system

$$A_h\mathbf{U} = \mathbf{F},$$

and specify what A_h, $\mathbf{U}$ and $\mathbf{F}$ are.

(ii) (7 points) Compute the quantity $\lim_{h\to 0+} \rho(A_h^{-1})$, where ρ is the spectral radius. You can use the following fact without proof: for the following tridiagonal matrix with $bc > 0$,

$$B = \begin{bmatrix} a & b & & & & \\ c & a & b & & & \\ & c & a & b & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & a & b \\ & & & & c & a \end{bmatrix} = \text{tridiag}\{c, a, b\} \in \mathbb{R}^{M\times M}$$

the eigenvalues are given by

$$\lambda_s = a + 2\sqrt{b}\sqrt{c}\cos\phi_s, \quad \phi_s = \frac{s\pi}{M+1}, \quad s = 1, \dots, M$$

and the corresponding eigenvectors are

$$x^{(s)} = \left[x_1^{(s)}, \dots, x_M^{(s)}\right]^\top, \quad x_k^{(s)} = \left(\frac{c}{b}\right)^{k/2} \sin(k\phi_s).$$

[Solution] (i) Consider equidistant grids on $[0, 1]$ with $M + 2$ points: $x_j := \frac{j}{M+1}$, $j = 0, ..., M+1$. Let $h = \frac{1}{M+1}$. Using the central difference, we have the following approximations:

$$u_x(x_j) \approx \frac{u(x_{j+1}) - u(x_{j-1})}{2h},$$

and

$$u_{xx}(x_j) \approx \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2},$$

for $j = 1, \dots, M$. With the above, we obtain:

$$A_h \mathbf{U} = \mathbf{F},$$

where

$$A_h = \frac{1}{h^2} \begin{bmatrix} -2 & 1+2h & & & & \\ 1-2h & -2 & 1+2h & & & \\ & 1-2h & -2 & 1+2h & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1-2h & -2 & 1+2h \\ & & & & 1-2h & -2 \end{bmatrix} \in \mathbb{R}^{M\times M},$$

$$\mathbf{U} \begin{bmatrix} U_1 \\ \vdots \\ U_M \end{bmatrix}, \quad \mathbf{F} := \begin{bmatrix} f_1 - \frac{1}{h^2} + \frac{4}{2h} \\ f_2 \\ \vdots \\ f_{M-1} \\ f_M - \frac{4}{h^2} - \frac{16}{2h} \end{bmatrix},$$

here, U_j approximates the value $u(x_j)$, and $f_j = f(x_j)$. The boundary condition is imposed by $U_0 = 1$ and $U_{M+1} = 4$.

(ii) Using the formula for eigenvalues of tridiagonal matrices, eigenvalues of A_h is given by

$$\lambda_s = \frac{-2 + 2\sqrt{1-4h^2}\ \cos(\pi s h)}{h^2}, \; s = 1, \dots, M.$$

Therefore,

$$\rho(A_h^{-1}) = \frac{1}{\min_s |\lambda_s|} = \frac{1}{|\lambda_1|}.$$

By expanding $\cos(\pi h)$ and $\sqrt{1-4h^2}$ around $h = 0$, we have

$$|\lambda_1| = 2\frac{1 - (1 - 2h^2 + \mathcal{O}(h^4))(1 - (\pi h)^2/2 + \mathcal{O}(h^4))}{h^2} \to \pi^2 + 4,$$

as $h \to 0$. Thus, $\lim_{h\to 0+} \rho(A_h^{-1}) = \dfrac{1}{\pi^2 + 4}$.

Problem 12 (15 points) Consider the following **boundary value problem** for the unknown function $u(x)$:

$$u_{xx} - 2u_x = f(x), \quad 0 < x < 1, \quad u(0) = 2, \quad u(1) = 1,$$

where $f(x) = \sin(\pi x)$.

(i) (8 points) Construct a finite difference method using central differences to approximate both u_{xx} and u_x with equidistant grid points on $[0, 1]$. In other words, obtain the discretized linear system

$$A_h \mathbf{U} = \mathbf{F},$$

and specify what A_h, $\mathbf{U}$ and $\mathbf{F}$ are.

(ii) (7 points) Compute the quantity $\lim_{h \to 0+} \rho(A_h^{-1})$, where ρ is the spectral radius. You can use the following fact without proof: for the following tridiagonal matrix with $bc > 0$,

$$A = \begin{bmatrix} a & b & & & & \\ c & a & b & & & \\ & c & a & b & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & a & b \\ & & & & c & a \end{bmatrix} = \text{tridiag}\{c, a, b\} \in \mathbb{R}^{M \times M}$$

the eigenvalues are given by

$$\lambda_s = a + 2\sqrt{b}\sqrt{c}\cos\phi_s, \quad \phi_s = \frac{s\pi}{M+1}, \quad s = 1, ..., M$$

and the corresponding eigenvectors are

$$x^{(s)} = \left[x_1^{(s)}, \ldots, x_M^{(s)}\right]^\top, \quad x_k^{(s)} = \left(\frac{c}{b}\right)^{k/2} \sin(k\phi_s).$$

[Solution] (i) Consider equidistant grids on $[0, 1]$ with $M + 2$ points: $x_j := \frac{j}{M+1}$, $j = 0, \cdots, M+1$. Let $h = \frac{1}{M+1}$. Using the central difference, we have the following approximations:

$$u_x(x_j) \approx \frac{u(x_{j+1}) - u(x_{j-1})}{2h},$$

and

$$u_{xx}(x_j) \approx \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2},$$

for $j = 1, \cdots, M$. With the above, we obtain:

$$A_h \mathbf{U} = \mathbf{F},$$

where

$$A_h = \frac{1}{h^2}\begin{bmatrix} -2 & 1-h & & & & \\ 1+h & -2 & 1-h & & & \\ & 1+h & -2 & 1-h & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1+h & -2 & 1-h \\ & & & & 1+h & -2 \end{bmatrix} \in \mathbb{R}^{M\times M},$$

$$\mathbf{U}\begin{bmatrix} U_1 \\ \vdots \\ U_M \end{bmatrix}, \quad \mathbf{F} := \begin{bmatrix} f_1 - \frac{2}{h^2} - \frac{4}{2h} \\ f_2 \\ \vdots \\ f_{M-1} \\ f_M - \frac{1}{h^2} + \frac{2}{2h} \end{bmatrix},$$

here, U_j approximates the value $u(x_j)$, and $f_j = f(x_j)$. The boundary condition is imposed by $U_0 = 2$ and $U_{M+1} = 1$.

(ii) Using the formula for eigenvalues of tridiagonal matrices, eigenvalues of A_h is given by

$$\lambda_s = \frac{-2 + 2\sqrt{1-h^2}\ \cos(\pi s h)}{h^2}, \ s = 1, ..., M.$$

Therefore,

$$\rho(A_h^{-1}) = \frac{1}{\min_s |\lambda_s|} = \frac{1}{|\lambda_1|}.$$

By expanding $\cos(\pi h)$ and $\sqrt{1-h^2}$ around $h = 0$, we have

$$|\lambda_1| = 2\frac{1 - (1 - h^2/2 + \mathcal{O}(h^4))(1 - (\pi h)^2/2 + \mathcal{O}(h^4))}{h^2} \to \pi^2 + 1,$$

as $h \to 0$. Thus, $\lim\limits_{h\to 0+} \rho(A_h^{-1}) = \dfrac{1}{\pi^2+1}$.