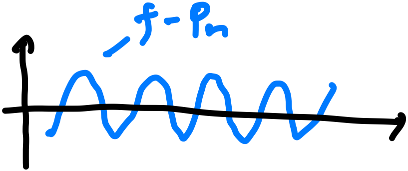
# Lecture 13 summary: Polynomial approximation

Süli–Mayers: Sections 8.1–8.5

1. Function approximation by polynomials
2. Weierstrass approximation theorem
3. Existence and uniqueness of the minimax polynomial
4. Equioscillation theorem



# Summary: function approximation

1. We want to construct a polynomial $p$ such that $\|f - p\|$ is small enough for some norm $\|\cdot\|$ and given function $f$.

2. Weierstrass approximation theorem: For $f \in C[a, b]$, and for any small $\varepsilon > 0$, there sxists a polynomial $p$ such that

$$\|f - p\| \leq \varepsilon.$$

   *This $p$ can be chosen from the set of all polynomials.

3. Minimax problem: Find such a polynomial $p \in P_n$:

$$\|f - p_n\|_\infty = \inf_{q \in P_n} \|f - q\|_\infty.$$

4. Existence and uniqueness (Thm 8.2 and Thm 8.5)
5. Oscillation: De la Vallée Poussin's theorem (8.3) and Equioscillation theorem (8.4)

# Lecture 14

## Chebyshev Polynomial

In general, it is hard to obtain simple, closed form of the minimax polynomial for a (general) given $f$.

However, if $f$ is a power of $x$ and we want to approximate it with a polynomial with lower degree, it is possible.

$[a, b] = [-1, 1]$

**Def** Chebyshev polynomial $T_n$ of degree $n$ is given by

$$T_n(x) := \cos(n \cos^{-1}(x)), \quad n = 0, 1, \cdots$$
$$x \in [-1, 1]$$

**Ex 1.**

$\underline{n=0}$ $T_0(x) = \cos(0) = 1.$ $\qquad \cos(2\theta) = 2\cos^2(\theta) - 1$

$\underline{n=1}$ $T_1(x) = \cos(\cos^{-1}(x)) = x$

$\underline{n=2}$ $T_2(x) = \cos(2\cos^{-1}(x))$

$= 2\cos^2(\cos^{-1}(x)) - 1 = 2x^2 - 1.$

## Properties.

In general,

$$\cos((n-1)\theta) + \cos((n+1)\theta) = 2\cos(\theta)\cos(n\theta)$$

and thus.....

(i) $T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$

for $n = 1, 2, 3 \dots$

(ii) For $n \geq 1$, the leading coefficient (the coefficient of the largest degree monomial) is $2^{n-1}$ for $T_n$.

(Follows from (i))

$T_n = 2^{n-1}x^n + 0x^{n-1} + \cdots$

(iii) When $n$ is even, $T_n$ is an even function

$n$ is odd, $T_n$ is an odd function

(This follows from (i))

(iv) zeros of $T_n$ for $n \geq 1$ are given by

$$x_j = \cos\left(\frac{(2j-1)\pi}{2n}\right) \quad j = 1, \ldots, n.$$

They are distinct over $[-1, 1]$.

(Like finding zeros of cos functions!)

(v) $|T_n(x)| \leq 1$ for all $x \in [-1, 1]$ and $x \geq 0$.

(From definition)

(vi) For $n \geq 1$, $T_n(x_k) = \pm 1$, alternately at points $x_k = \cos\left(\frac{k\cancel{x}}{n}\right)$, $k = 0, 1, \cdots, n$,

(proof: $T_n(x_k) = \cos(n \cdot \underbrace{\cos^{-1}\left(\cos\left(\frac{k\cancel{x}}{n}\right)\right)}_{\frac{k\cancel{x}\pi}{n}})$

$= \cos(k\pi) = (-1)^k$ ).

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Thm 8.6 (best approx. to $x^{n+1}$ on $P_n$).

The following poly. $P_n \in P_n$ is the minimax poly. of $f(x) = x^{n+1}$ :

$$P_n = x^{n+1} - 2^{-n} T_{n+1}(x)$$

on $[-1, 1]$

Proof) From property (ii)

$\left( T_{n+1}(x) = 2^n x^{n+1} + 0\, x^n + \cdots \right)$

$x^{n+1} + \cancel{\mathbb{0}}\, x^n + \cdots$

$P_n \in P_n$.

Also, the difference $x^{n-1} - P_n = 2^{-n} T_{n+1}(x)$

satisfies $|x^{n-1} - P_n| \le 2^{-n}$ for any $x \in [-1,1]$

by (v).

By (vi), $|x_j^{n-1} - P_n(x_j)| = 2^{-n}$

$x_j = \cos\left(\frac{j\pi}{n}\right)$, for $j = 0, \cdots, n$.

Therefore, by the equioscillation thm,

$P_n$ is the unique minimax poly.

from $P_n$ to approximate $f(x) = x^{n+1}$. $\blacksquare$

Remark) when the leading coefficient

is 1, that polynomial is said to be

a monic polynomial. $2^{-n+1} T_n(x)$ is a monic

polynomial, for example.

Corollary 8.1 For $n \geq 0$, among all monic polynomial of degree $n+1$, $2^{-n} T_{n+1}(x)$ and $-(2^{-n} T_{n+1}(x))$ have the smallest $\infty$-norm on $[-1, 1]$.

Proof) $\min_{\substack{r \in P_{n+1} \\ r: \text{monic}}} \|r\|_\infty = \min_{q \in P_n} \|x^{n+1} - q\|_\infty$

(From thm 8.6, we know the minimax poly for $x^{n+1}$ is : $x^{n+1} - 2^{-n} T_{n+1}$)

$= \|x^{n+1} - (x^{n+1} - 2^{-n} T_{n+1})\|_\infty$

$= \|2^{-n} T_{n+1}\|_\infty \quad (= 2^{-n}$

Thus, $2^{-n} T_{n+1}$ achieves the minimum $L^\infty$-norm. The same argument holds for $-(2^{-n} T_{n+1})$.

$\blacksquare$

# Interpolation revisited

Recall the Lagrane interpolation problem,

$$\left( P_n(x_k) = f(x_k) \quad x_k = 0, \cdots, n \right)$$

where the interpolation error was given by

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)\,!} \Pi_{n+1}(x)$$

$$\Pi_{n+1}(x) = (x - x_0) \cdots (x - x_n),$$

for some $\xi \in (a, b)$,

and for a function $f$ which is $(n+1)$-times continuously differentiable on $[a, b]$.

Corollary 8.1 suggests that if we can take intepolation points $x_0, x_1, \cdots, x_n$ such that $\Pi_{n+1}$ corresponds to $2^{-n} T_{n+1}$, We can expect small error for

$\|f - P_n\|_\infty$.

**Notice** (scaling $[-1,1]$ to $[a,b]$)

We defined Chebyshev polynomial on $[-1,1]$ and we can extend the results to a general interval $[a,b]$ by linear scaling

$$\underset{[-1,1]}{x} \longrightarrow \underset{[a,b]}{y} : y = \frac{(b-a)}{2}(x+1) + a$$

i.e. the Chebyshev polynomial on $[a,b]$ looks like...

$$T_n^{[a,b]}(y) = T_n\Big(\underbrace{\frac{2}{b-a}(y-a) - 1}_{t(y):\,[a,b] \to [-1,1]}\Big)$$

**Thm 8.7**

Suppose that $f$ is $n+1$ times cont. differentiable on $[a,b]$. With interpolation

Points $x_j = \frac{1}{2}(b-a)\cos\left(\frac{(j+\frac{1}{2})}{n+1}\pi\right) + \frac{1}{2}(b+a)$

for $j = 0, 1, \dots, n$

(zeros of scaled Chebyshev polynomials on $[a, b]$).

The Lagrange interpolation polynomial $P_n \in \mathbb{P}_n$ of $f$, have the following error bound:

$$\|f - P_n\|_\infty \le \frac{(b-a)^{n+1}}{2^{(2n+1)}(n+1)!} M_{n+1}.$$

where $M_{n+1} = \max_{\eta \in [a,b]} |f^{(n+1)}(\eta)|$

Proof] $\|f - P_n\|_\infty \le \frac{M_{n+1}}{(n+1)!} \|\Pi_{n+1}\|_\infty$ and.

$$\|\Pi_{n+1}\|_\infty = \|(x-x_0)\cdot(x-x_1)\cdots(x-x_n)\|_\infty$$

$(x - x_i) = \left(\frac{b-a}{2}\right)\cdot[t(x) - y_i)$

where $t(x) \in [-1,1]$ and $y_i$'s are zeros of $T_n$.

$$= \left(\frac{b-a}{2}\right)^{n+1} \left\| \prod_{i=0}^{n} (t(x) - y_i) \right\|$$

n+1 degree monic polynomial with zeros of $T_{n+1}$ is nothing but $2^{-n} T_{n+1}$.

$$= \left(\frac{b-a}{2}\right)^{n+1} 2^{-n} \| T_n(t(x)) \|_\infty$$

$$\underset{(v)}{=} \left(\frac{b-a}{2}\right)^{n+1} 2^{-n} = \frac{(b-a)^{n+1}}{2^{2n+1}}$$

This proves the claim. $\boxtimes$

Remark) Interpolation Polynomial $P_n$ using the Chebyshev nodes (zeros of $T_n$) is not minimax polynomial in general.

(minimax poly. depends on the given $f$. but the Chebyshev nodes don't depend on $f$ at all)

→ Animation. (Runge phenomenon doesn't happen for Chebyshev nodes.)

Some known results on interpolation . . . . .
(without proof)

Thm (Faber)

For any prescribed sequence of nodes,

$a \le x_0 < x_1 < \cdots < x_n \le b$

there exists a function $f \in C[a,b]$
such that the interpolation poly $P_n$ doesn't converge uniformly to $f$

(existence of "fooling function" for any nodes)

Thm

For any $f \in C[a,b]$, there exists sequence of nodes

$a \le x_0 < x_1 < \cdots < x_n \le b$ such that the interpolation poly. $P_n$ to $f$ satisfies

$$\lim_{n \to \infty} \max_{x \in [a,b]} |f(x) - P_n(x)| = 0.$$

(Existence of nodes for uniform convergence)

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# Weighted $L^2$ space and orthogonal polynomials

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$L^2_w$ space ...

<u>orthogonality</u>

For some "basis function"

$$\langle \varphi_i, \varphi_j \rangle_{L^2_w} = \int_a^b \varphi_i(x) \cdot \overline{\varphi_j(x)} \cdot w(x)\,dx = \begin{cases} c & i = j \\ 0 & i \neq j \end{cases}$$

In linear algebra ... $\mathbb{R}^n$

<u>orthogonality</u> basis $\vec{e}_i \in \mathbb{R}^n$

$$\underbrace{\vec{e}_i \cdot \vec{e}_j}_{\text{inner product.}} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

## Representation.

For any function $f \in L^2_w$

$\left( L^2_w := \left\{ f \;\middle|\; \int_a^b |f(x)|^2 w(x) dx < \infty \right\} \right)$

$$f(x) \overset{``}{=}{}^{"} \frac{1}{C} \underbrace{\sum_{n=0}^{\infty} \langle f, \varphi_n \rangle \varphi_n}_{S_n}$$

## Representation.

For any vector $\vec{a} \in \mathbb{R}^n$.

we can express

$$\vec{a} = (\vec{a}\cdot\vec{e}_1)\vec{e}_1 + (\vec{a}\cdot\vec{e}_2)\vec{e}_2 + \cdots + (\vec{a}\cdot\vec{e}_n)\vec{e}_n$$

This property is called completeness.

This requires just more than orthogonality.

and this "=" is in the sense

$$\lim_{n\to\infty} \left( \left( \int_a^b |f(x) - S_n(x)|^2 w(x) dx \right)^{\frac{1}{2}} \right) = 0$$

"$S_n$ converges to $f$ in $L^2_w$."

## Examples.

i) Chebyshev Polynomials $([a, b] = [-1, 1])$

is known to be orthogonal with $w(x) = \frac{1}{\sqrt{1-x^2}}$.

$$\langle T_i, T_j \rangle = \int_{-1}^{1} \cos(i\cos^{-1}(x)) \cos(j\cos^{-1}(x)) \cdot \frac{1}{\sqrt{1-x^2}}\, dx$$

$t = \cos^{-1}(x)$

$$\frac{dt}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$= \int_0^{\pi} \cos(it)\cos(jt)$$

$$= \frac{1}{2}\int_0^{\pi} \left[\cos((i+j)t) + \cos((i-j)t)\right] dt$$

$$= \begin{cases} \frac{\pi}{2} & i = j \\ 0 & i \neq j \end{cases}.$$

Thus, Chebyshev Poly. are orthogonal.

As a fact, they are complete.