MA2501, Spring 2020, Numerical Methods

March 2020

Problem 1

Consider $f \in C^{(4)}([-a, a])$ and let $p_3(x)$ be the interpolation polynomial of degree 3 satisfying

$$p_3(-a) = f(-a), \quad p_3(a) = f(a), \quad p_3'(-a) = f'(-a), \quad p_3'(a) = f'(a).$$

Show that if $M_4 = \max_{-a \le x \le a} |f^{(4)}(x)|$, then

$$|f(x) - p_3(x)| \le \frac{a^4}{24} M_4.$$

Guidelines: Use theorem about the error of Hermite interpolation (page 190 in Süli and Mayers).

Problem 2

A quadrature formula on the interval $[-1, 1]$ uses the quadrature points $x_0 = -\alpha$ and $x_1 = \alpha$, where $0 < \alpha \le 1$:

$$\int_{-1}^{1} f(x)\, dx \approx w_0 f(-\alpha) + w_1 f(\alpha).$$

The formula is required to be exact whenever f is a polynomial of degree 1.

- Show that $w_0 = w_1 = 1$, independent on the value of α.
- Show also that there is one particular value of α for which the formula is exact also for polynomials of degree 2, and show that for this value the formula is exact also for all polynomials of degree 3.

Problem 3

The Newton-Cotes formula with $n = 3$ on the interval $[-1, 1]$ is

$$\int_{-1}^{1} f(x)\, dx \approx w_0 f(-1) + w_1 f(-1/3) + w_2 f(1/3) + w_3 f(1).$$

Using the fact that this formula is to be exact for all polynomials of degree 3, or otherwise, show that

$$\begin{aligned} 2w_0 + 2w_1 &= 2 \\ 2w_0 + \frac{2}{9} w_2 &= \frac{2}{3}, \end{aligned}$$

and find the values of the weights w_0, w_1, w_2 and w_3.

Problem 4

Write down the errors in the approximation of

$$\int_0^1 x^4 dx \quad \text{and} \quad \int_0^1 x^5 dx$$

by the trapezium rule and the Simpson's rule (page 202 and 203 in the textbook). Use the exact values of the two integrals. Hence find the value of the constant C for which the trapezium rule gives the correct result for the calculation of

$$\int_0^1 (x^5 - Cx^4) dx,$$

and show that the trapezium rule gives a more accurate result than the Simpson's rule when $\frac{15}{14} < C < \frac{85}{74}$.

Problem 5

With the usual notation for the Newton-Cotes quadrature formula and using the equally spaced quadrature points $x_k = a + k\,h$ for $k = 0, 1, \ldots, n$ and $n \geq 1$, show that the quadrature weights are such that $w_k = w_{n-k}$ for $k = 0, \ldots, n$.