MA2501, Spring 2020, Numerical Methods

February 2020

Problem 1

Let $f \in C[a,b]$. Consider the 2-norm:

$$\|f\|_2 := \left(\int_a^b w(x)\, |f(x)|^2 \, dx \right)^{\frac{1}{2}}$$

where w is a given function which is defined, real, continuous, positive and integrable on $[a,b]$.

Prove that

1. $\|f\|_2 \le W \|f\|_\infty$ with $W := \left(\int_a^b w(x)\, dx \right)^{\frac{1}{2}}$
2. $\forall \varepsilon > 0$ (however small) and $M > 0$ (however large) constants, $\exists\, f \in C[a,b]$ such that
$$\|f\|_2 < \varepsilon, \qquad \|f\|_\infty > M.$$

Problem 2

The Newton form of the interpolation polynomial on distinct nodes $x_0, \ldots, x_n$ is

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1}). \quad (1)$$

The divided differences (DD) are defined as follows:

DD of order 1 $\quad f[x_0] := f(x_0)$

DD of order 2 $\quad f[x_0, x_1] := \frac{f[x_1]-f[x_0]}{x_1-x_0}$

$\vdots \qquad \vdots$

DD of order k $\quad f[x_0, x_1, \ldots, x_k] := \frac{f[x_1,\ldots,x_k]-f[x_0,\ldots,x_{k-1}]}{x_k-x_0}$

Prove the following results.

- **Theorem**: In the Newton form of the interpolation polynomial (1)

$$a_k := f[x_0, \ldots, x_k], \qquad k = 0, \ldots, n.$$

  Guidelines: The proof is by induction.

  - For $k = 0$ $f(x_0) = p(x_0) = a_0$.
  - Assume $a_i := f[x_0, \ldots, x_i]$, $i = 0, \ldots, k-1$ are the coefficients of the Newton interpolation polynomial of degree k (induction hypothesis). Prove then that $a_k := f[x_0, \ldots, x_k]$ is the missing coefficient for the polynomial of degree $k+1$.

    To prove this consider the Newton interpolation polynomial p_k of degree k (i.e. as in (1) with $n = k$) and evaluate it in x_k.
  - Using the indusction hypothesis, you should be able to deduce that

$$f[x_0, x_k] = \frac{f(x_k) - a_0}{x_k - x_0},$$

    then

$$f[x_0, x_1, x_k] = \frac{f[x_0, x_k] - a_1}{x_k - x_1},$$

    and so repeating the procedure until

$$f[x_0, x_1, \ldots, x_{k-1}, x_k] = \frac{f[x_0, \ldots, x_{k-2}, x_k] - a_{k-1}}{x_k - x_{k-1}} = a_k,$$

    which is what you want to prove. This will be enough to conclude the induction proof.

- **Theorem**: Let p_n be the interpolation polynomial on the distinct nodes $x_0, \ldots, x_n$. Assume $t \neq x_i$, for $i = 0, \ldots, n$, then

$$f(t) - p_n(t) = f[x_0, \ldots, x_n, t] \prod_{j=0}^{n} (t - x_j).$$

Guidelines: It follows directly by considering the polynomial q interpolating f on the nodes $x_0, \dots, x_n, t$ in Newton form.

Problem 3

- Consider the following interpolation problem: find the polynomial $p \in \Pi_3$ satisfying

$$p_3(x_i) = f(x_i), \quad p_3'(x_i) = f'(x_i), \qquad i = 0, 1.$$

Here f is a continuous and differentiable function on the interval $[a, b]$ containing the nodes x_0 and x_1. Find p_3 assuming it has the form

$$p_3(x) = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^2(x - x_1). \qquad (2)$$

You need to determine the coefficients a, b, c, d using the values of f and its derivatives in x_0 and x_1.

- **Hermite interpolation on one node and with several derivatives.** Find the Hermite interpolation polynomial satisfying

$$p_n(x_0) = f(x_0), \quad p_n^{(k)}(x_0) = f^{(k)}(x_0)$$

for $k = 1, \dots, n$. Here the superscript $^{(k)}$ denotes the k-derivative. What do you obtain?

Guidelines: modify the format (2) in an appropriate way so that it fits your problem. Then find the coefficients of the polynomial.