Numerical ordinary differential equations: Consistency and convergence of one-step methods

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We consider the initial value problem

$$\begin{cases} \dot{y}(t) & = & f(t, y(t)) \\ y(a) & = & y_0 \end{cases} \qquad t \in [a, b],$$

and where $y(t) \in \mathbb{R}^m$ $y(t) = [y_1(t), \ldots, y_m(t)]^T$ and $y_i(t) \in \mathbb{R}$.

Definition

Given an initial value problem on $[a, b]$, suppose

$$t_0 = a < t_1 < \cdots < t_N = b,$$

and

$$y_1 \approx y(t_1), \ldots, y_N \approx y(t_N),$$

are the approximations obtained by a numerical one-step method with $h = \frac{b-a}{N}$. We say that the method is **convergent** if and only if

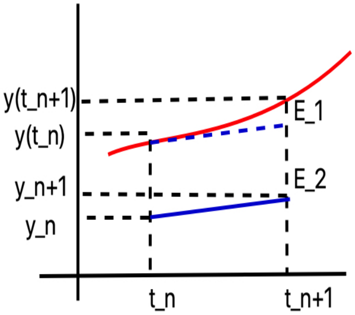
$$\lim_{h\to 0, N\to\infty} \|y(b) - y_N\| = 0.$$

Example Euler method

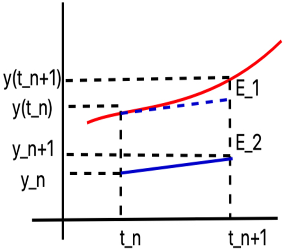
The error contribution over one step of a one-step method $e_{n+1} = y(t_{n+1}) - y_{n+1}$ is the sum of two terms

$$e_{n+1} = E_1 + E_2,$$

- E_1 error of the method as applied to the exact solution $y(t_n)$ at time t_n
- E_2 error due to earlier approximations and carried over from step t_n



Error at time t_{n+1} as the sum of two terms



Let z_{n+1} one step of Euler method starting from $y(t_n)$:

$$z_{n+1} = y(t_n) + h\, f(t_n, y(t_n))$$

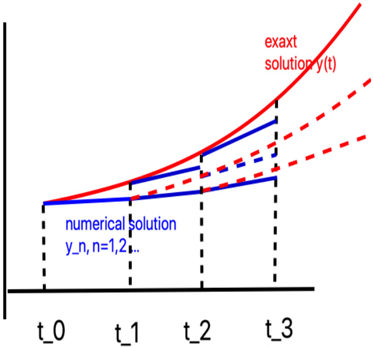
then

$$\begin{aligned} y(t_{n+1}) - y_{n+1} &= y(t_{n+1}) - z_{n+1} + z_{n+1} - y_{n+1} \\ y(t_{n+1}) - y_{n+1} &= E_1 + E_2 \end{aligned}$$

where

$$E_1 := y(t_{n+1}) - z_{n+1}, \qquad E_2 := z_{n+1} - y_{n+1}.$$

Accumulation of error over several steps



In the convergence analysis, we are interested in the difference between the red curve (exact solution) and the blue picewise-linear curve (numerical solution) at al times t_n.
The method applied to the exact solution $y(t_n)$ at time t_n over one step gives z_{n+1} which are here represented by the blue tangents to the red curve.
But there is another contribution to the error that comes from the fact that the method is applied to y_n at step t_n and not to $y(t_n)$. At time t_0 this error is zero.

Local truncation error

Definition

The **local truncation error** for a one-step method

$$y_{n+1} = \Phi_h(y_n)$$

is

$$\sigma(t_{n+1}, h) = y(t_{n+1}) - z_{n+1}$$

where $z_{n+1} = \Phi_h(y(t_n))$.

Definition

We say that the method is **consistent** if and only if

$$\lim_{h\to 0} \frac{\sigma(t_{n+1}, h)}{h} = 0, \qquad \text{for } n = 0, \ldots, N-1, \quad h = \frac{b-a}{N}.$$

Definition

A one-step method has order of convergence p if and only if

$$\sigma(t, h) = \mathcal{O}(h^{p+1}).$$

Proposition

The Euler method is consistent with order of convergence $p = 1$.

Proof (Using Taylor theorem):

$$
\begin{aligned}
\sigma(t_{n+1}, h) &= y(t_{n+1}) - z_{n+1} \\
&= y(t_n) + h\dot{y}(t_n) + \frac{h^2}{2}\ddot{y}(\gamma) - \quad \gamma \in (t_n, t_{n+1}) \\
&\quad -y(t_n) - h\, f(t_n, y(t_n)) \\
&= \frac{h^2}{2}\ddot{y}(\gamma).
\end{aligned}
$$

Lemma: If $h = \frac{b-a}{N}$, $L > 0$ $D \geq 0$ and s_n, $n = 0, 1, \ldots, N-1$ is such that s_n is positive and

$$s_{n+1} \leq (1 + hL)s_n + D, \quad \text{for } n = 0, \ldots, N-1,$$

then

$$s_m \leq (1 + hL)^N s_0 + D\frac{(1 + hL)^N - 1}{hL}, \quad \forall m \leq N$$

Proof: from the hypothesis

$$s_n \leq (1 + hL)s_{n-1} + D, \quad \text{for } n = 1, \ldots, N,$$

and $1 + hL > 0$ so by induction

$$s_n \leq (1 + hL)^n s_0 + \left[(1 + hL)^{n-1} + \cdots + 1\right] D, \quad \text{for } n = 1, \ldots, N.$$

Since for $\alpha > 0$, $\alpha^N + \alpha^{N-1} + \cdots + 1 = \frac{\alpha^N - 1}{\alpha - 1}$, here with $\alpha = 1 + hL$ we get

$$s_n \leq (1 + hL)^N + D\frac{(1 + hL)^N - 1}{hL}.$$

End of the proof

Corollary
Under the same hypothesis of Lemma

$$s_m \leq e^{L(b-a)}s_0 + D\frac{e^{L(b-a)}-1}{hL}, \quad \forall m \leq N$$

Proof
From the result of Lemma, since

$$1 + hL \leq e^{hL}$$

so

$$(1 + hL)^N \leq e^{hLN}$$

where $hN = (b - a)$.
End of the proof

Convergence of the Euler method

Theorem. Assume $f(t, y)$ is continuous in t and y, and satisfies the Lipschitz condition with respect to y on $[a, b] \times \mathbf{R}^m$, y sufficiently smooth, then the Euler method converges.

Proof: $e_N := y(t_N) - y_N$ we want to prove that $\lim_{h\to 0, N\to\infty} \|e_N\| = 0$. Consider $z_{n+1} = y(t_n) + h\, f(t_n, y(t_n))$ then

$$\|y(t_{n+1}) - y_{n+1}\| \leq \|y(t_{n+1}) - z_{n+1}\| + \|z_{n+1} - y_{n+1}\|$$

Let $e_i := y(t_i) - y_i$, so $e_{n+1} = y(t_{n+1}) - y_{n+1}$ then

$$\begin{aligned}
\|e_{n+1}\| &\leq \|\sigma(t_{n+1}, h)\| + \|y(t_n) + h\, f(t_n, y(t_n)) - y_n - h\, f(t_n, y_n)\| \leq \\
&\leq \|\sigma(t_{n+1}, h)\| + \|e_n\| + h\|f(t_n, y(t_n)) - f(t_n, y_n)\| \\
&\leq \|\sigma(t_{n+1}, h)\| + \|e_n\| + hL\|e_n\| = \|\sigma(t_{n+1}, h)\| + (1 + hL)\|e_n\|
\end{aligned}$$

Using the Corollary with $s_n := \|e_n\|$ and $D = \max_{a\leq t\leq b} \|\sigma(t, h)\|$ we get

$$\|e_m\| \leq e^{L(b-a)}\|e_0\| + D\,\frac{e^{L(b-a)} - 1}{hL}, \quad \forall m \leq N$$

with $\|e_0\| = \|y(t_0) - y_0\| = 0$.

Proof continues

Because of the consistency of Euler's method

$$D = \max_{a \le t \le b} \|\sigma(t,h)\| \le \frac{h^2}{2} \max_{a \le t \le b} \|\ddot{y}(t)\| \le C\ h^2$$

and so

$$\|e_N\| \le C\, h^2\, \frac{e^{L(b-a)} - 1}{hL} = \frac{C}{L}\left(e^{L(b-a)} - 1\right)\, h$$

so

$$\lim_{h \to 0} \|e_N\| = 0$$

End of the proof