Interpolation summary

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Interpolation problem

given the data

x	x_0	x_1	$\dots$	x_n
y	y_0	y_1	$\dots$	y_n

$x_0, \dots, x_n$ distinct FIND the polynomial $p_n \in \Pi_n$ such that

$$p_n(x_i) = y_i, \qquad i = 0, 1, \dots, n.$$

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EXISTENCE: **Lagrange form of the interpolation polynomial**

$$p_n(x) = \sum_{i=0}^{n} \ell_i(x)\, y_i, \qquad \ell_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}.$$

where ℓ_i are the Lagrangian basis functions. Also UNIQUENESS!

Interpolation error

(Theorem 6.2): $n \geq 0$, f real-valued, defined, continuous on $[a, b]$, $f^{(n+1)}$ exists and is continuous on $[a, b]$.
Then given $x \in [a, b]$ $\exists$ $\xi = \xi(x) \in (a, b)$ s.t.

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{j=0}^{n} (x - x_j),$$

Error bound

$$\max_{x \in [a,b]} |f(x) - p_n(x)| \leq \frac{M_{n+1}}{(n+1)!} \max_{x \in [a,b]} |\omega(x)|, \quad M_{n+1} = \max_{\xi \in [a,b]} |f^{(n+1)}(\xi)|$$

call

$$\omega(x) = \prod_{j=0}^{n} (x - x_j).$$

DEF:We have convergence of p_n uniformly to f if

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When

$$\lim_{n\to\infty} M_{n+1} \max_{x\in[a,b]} |\omega(x)|$$

goes to ∞ faster than

$$\frac{1}{(n+1)!}$$

goes to zero, the interpolation does not converge!

Runge function

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BUT with a special choice of nodes one can get much better results for this function: these are the **Chebishev nodes**.

Theorem (Faber)
For any prescribed system of nodes

$$a \leq x_0^n < x_1^n < \cdots < x_n^n \leq b, \quad n \geq 0 \tag{1}$$

there exists a continuous function f on $[a, b]$ such that the interpolating polynomial p_n does not converge uniformly to f.

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Theorem
If f is a continuous function on $[a, b]$ there exists a system of nodes (1) such that the interpolation polynomials p_n to f satisfy

$$\lim_{n\to\infty} \max_{x\in[a,b]} |f(x) - p_n(x)| = 0.$$

DEF A linear space V (over $\mathbf{R}$) with a norm $\|\cdot\| : V \to \mathbf{R}$ is called a **normed linear space**.

The norm function $\|\cdot\| : V \to \mathbf{R}$ must satisfy the three norm axioms we are already familiar with:

1. $\|f\| = 0$ if and only if $f = 0$ in V;
2. $\|\lambda f\| = \lambda \|f\|$ for all $\lambda \in \mathbf{R}$, and all $f \in V$;
3. $\|f + g\| \leq \|f\| + \|g\|$ for all f and g in V.

See example 8.1 and 8.1 in the book and also Lemma 8.1.

Weiestrass approximation theorem

Theorem

Let f be a real-valued function, defined on $[a, b]$ and continuous on $[a, b]$. Then for all $\varepsilon > 0$ there exist a polynomial p such that

$$\|f - p\|_\infty < \varepsilon$$

Analogous result holds in the 2-norm.

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*P*roof: based on the use of Bernstein polynomials which are

$$p_{n,k}(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k}, \quad x \in [0,1].$$

We can look at the case $[a,b]=[0,1]$ (then use a linear transformation).

One can show that the polynomial p_n

$$p_n(x) = \sum_{k=0}^{n} p_{n,k}(x) f(\frac{k}{n}), \quad x \in [0,1]$$

for $n = n(\varepsilon)$ is such that $\|f-p_n\|_\infty < \varepsilon$.

Best approximation problem: given $f \in C[a,b]$ $n \geq 0$ find $p_n \in \Pi_n$ such that

$$\|f - p_n\|_\infty = \inf_{q \in \Pi_n} \|f - q\|_\infty,$$

if such p_n exists it is called **polynomial of best approximation of degree** n **to** f **in** $\|\cdot\|_\infty$ **or minimax polynomial**.

Theorem 8.2 and 8.5
Let f be in $C[a,b]$ then there exist one and only one $p_n \in \Pi_n$ such that

$$\|f - p_n\|_\infty = \min_{q \in \Pi_n} \|f - q\|_\infty.$$

Consider Definition 8.2 and Lemma 8.2.
DEF Monic polynomial. Consider

$$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$$

a polynomial of degree n, p_n is called monic if $a_n = 1$.

Example

- $\omega(x)$ is monic;
- $2^{1-n}\, T_n(x)$ is monic.

Theorem If p is a monic polynomial of degree n on $[-1,1]$ then

$$\|p\|_\infty = \max_{-1\leq x\leq 1} |p(x)| \geq 2^{1-n}.$$

Proof. By contradiction. Assume $|p(x)| < 2^{1-n}$ for all x such that $|x| \leq 1$. Consider

$$q(x) = 2^{1-n}\, T_n(x),$$

and the points $x_i = \cos(\frac{i\pi}{n})$, $i = 0, 1, \ldots, n$ where from lemma 8.2 we have $q(x_i) = (-1)^i\, 2^{1-n}$. We have

$$(-1)^i p(x_i) \leq |p(x)| < 2^{1-n} = (-1)^i q(x_i),$$

leading to

$$0 < (-1)^i (q(x_i) - p(x_i)), \quad i = 0, 1, \ldots, n.$$

So

- $q - p$ oscillates $n+1$ times in $[-1,1]$;
- $q - p$ has at least n zeros in $[-1,1]$;
- but $q - p$ has degree at most $n-1$;

this implies $q - p$ is identically zero on $[-1,1]$ and so $q = p$, but $|q(x_i)| = 2^{1-n}$ and not $< 2^{1-n}$ as assumed for p, therefore this is a contradiction.

Remark: $q(x) = 2^{1-n}T_n(x)$ is a monic polynomial with norm $\|q\|_\infty = 2^{1-n}$ on $[-1, 1]$. This is the minimum possible attainable value of $\|p\|_\infty$ for any monic polynomial p of degree n on $[-1, 1]$.

Recall the **interpolation error bound:**

$$\max_{x\in[a,b]} |f(x) - p_n(x)| \leq \frac{M_{n+1}}{(n+1)!} \max_{x\in[a,b]} |\omega(x)|, \tag{2}$$

where

$$\omega(x) = \prod_{i=0}^{n}(x - x_i)$$

and where x_i are the interpolation nodes, and ω is a monic polynomial.

Choosing $x_0, \dots, x_n$ to be the zeros of the Chebishev polynomial $T_{n+1}(x)$ (i.e. the values $x_i = \cos\left(\frac{2i+1}{2n+2}\pi\right)$, see lemma 8.2) is just equivalent to letting

$$\omega(x) = 2^n T_{n+1}(x).$$

So this choice of interpolation nodes minimizes the norm $\|\omega\|_\infty$ on $[-1, 1]$, and optimizes the choice of interpolation nodes in the bound (2), regardless of the features of the function f.

Remark: The interpolation polynomial p_n obtained using the Chebishev nodes is NOT the minimax polynomial.