MA2501, Spring 2019, Numerical Methods

March 4th 2019

Problem 1

Consider $f \in C^{(4)}([-a,a])$ and let $p_3(x)$ be the interpolation polynomial of degree 3 satisfying

$$p_3(-a) = f(-a), \quad p_3(a) = f(a), \quad p_3'(-a) = f'(-a), \quad p_3'(a) = f'(a).$$

Show that if $M_4 = \max\limits_{-a \le x \le a} |f^{(4)}(x)|$, then

$$|f(x) - p_3(x)| \le \frac{a^4}{24} M_4.$$

Guidelines: Use theorem about the error of Hermite interpolation (page 190 in Süli and Mayers).

Solution This is Hermite interpolation with $n = 1$, from the theorem about the error of Hermite interpolation (page 190 in Süli and Mayers) we see that the exact expression for the error is

$$f(x) - p_3(x) = \frac{f^4(\xi)}{4!}[(x-a)(x+a)]^2.$$

We get

$$|f(x) - p_3(x)| \le \frac{M_4}{24} \max_{x \in [-a,a]} [x^2 - a^2]^2$$

finding the maximum on $[-a,a]$ of the polynomial $[x^2 - a^2]^2$ we obtain the result.

Problem 2

A quadrature formula on the interval $[-1, 1]$ uses the quadrature points $x_0 = -\alpha$ and $x_1 = \alpha$, where $0 < \alpha \leq 1$:

$$\int_{-1}^{1} f(x)\, dx \approx w_0 f(-\alpha) + w_1 f(\alpha).$$

The formula is required to be exact whenever f is a polynomial of degree 1.

- Show that $w_0 = w_1 = 1$, independent on the value of α.
- Show also that there is one particular value of α for which the formula is exact also for polynomials of degree 2, and show that for this value the formula is exact also for all polynomials of degree 3.

Problem 3

The Newton-Cotes formula with $n = 3$ on the interval $[-1, 1]$ is

$$\int_{-1}^{1} f(x)\, dx \approx w_0 f(-1) + w_1 f(-1/3) + w_2 f(1/3) + w_3 f(1).$$

Using the fact that this formula is to be exact for all polynomials of degree 3, or otherwise, show that

$$\begin{aligned} 2w_0 + 2w_1 &= 2 \\ 2w_0 + \frac{2}{9}w_2 &= \frac{2}{3}, \end{aligned}$$

and find the values of the weights w_0, w_1, w_2 and w_3.

Problem 4

Write down the errors in the approximation of

$$\int_0^1 x^4 dx \quad \text{and} \quad \int_0^1 x^5 dx$$

by the trapezium rule and the Simpson's rule (page 202 and 203 in the textbook). Use the exact values of the two integrals. Hence find the value of the constant C for which the trapezium rule gives the correct result for the calculation of

$$\int_0^1 (x^5 - Cx^4)dx,$$

and show that the trapezium rule gives a more accurate result than the Simpson's rule when $\frac{15}{14} < C < \frac{85}{74}$.

Solution The values of the two integrals are respectively 1/5 and 1/6. If we approximate both the two integrals with the trapezium rule we get in both cases the value 1/2 as approximation. So for the trapezium rule we get the two errors

$$|\frac{1}{5} - \frac{1}{2}|, \quad |\frac{1}{6} - \frac{1}{2}|,$$

and one proceeds similarly for the Simpson rule.

Let us denote with I the exact value of the integral

$$I = \int_0^1 (x^5 - Cx^4)dx = \frac{5 - 6C}{30},$$

similarly we will denote with T the approximation due to the trapezium rule and S the one due to the Simpson rule, and we have

$$I \approx T = \frac{1}{2} - C\frac{1}{2}, \qquad I \approx S = \frac{1}{6}\left(\frac{9 - 10C}{8}\right).$$

We obtain the following error functions

$$I - T = \frac{-10 + 9C}{30}, \qquad I - S = \frac{-5 + 2C}{240},$$

and the trapezium formula gives the exact value of the integral when $C = \frac{10}{9}$.

Both $I - T$ and $I - S$ are linear functions of C. We have to find the values of C such that $|I - T| \leq |I - S|$.

The two functions are plotted in figure 1: $|I - T|$ as a function of C decreases for values of $C \leq \frac{10}{9}$, and increases for $C > \frac{10}{9}$. $|I - S|$ has a similar behaviour, and is zero in $C = \frac{5}{2}$. It suffices to find the points of intersection of the two graphs. It turns out that the graph of $|I - T|$ intersects $|I - S| = S - I$ for $C < \frac{5}{2}$, and $S - I$ coincides with the line through the two points $(-5/240, 0)$ and $(5/2, 0)$ for $C < \frac{5}{2}$.

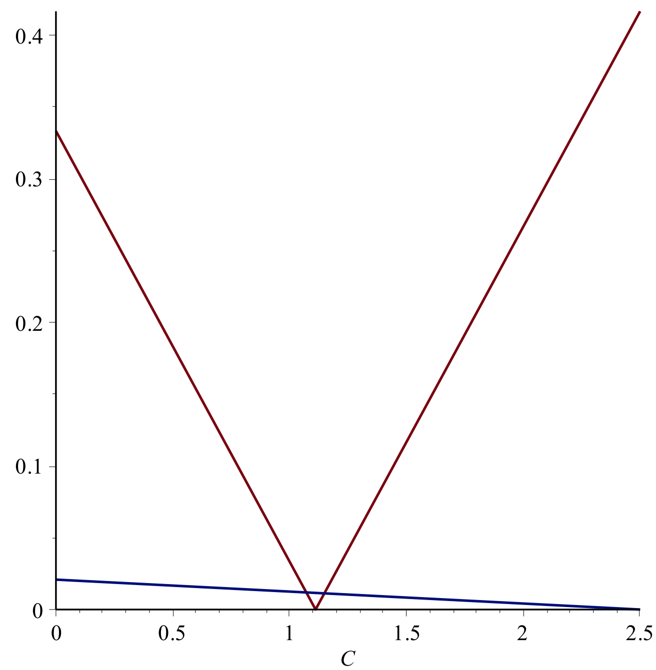

Figure 1: The two functions $|I - T|$ (in red) and $|I - S|$ (in blue) as functions of C.

Problem 5

With the usual notation for the Newton-Cotes quadrature formula and using the equally spaced quadrature points $x_k = a + k\,h$ for $k = 0, 1, \dots, n$ and $n \geq 1$, show that the quadrature weights are such that $w_k = w_{n-k}$ for $k = 0, \dots, n$.