MA2501, Spring 2019, Numerical Methods

February 17th 2019

Problem 1

Let $f \in C[a,b]$. Consider the 2-norm:

$$\|f\|_2 := \left(\int_a^b w(x)\, |f(x)|^2 \, dx \right)^{\frac{1}{2}}$$

where w is a given function which is defined, real, continuous, positive and integrable on $[a,b]$.

Prove that

1. $\|f\|_2 \leq W \|f\|_\infty$ with $W := \left(\int_a^b w(x)\, dx \right)^{\frac{1}{2}}$

   Solution:

   $$\|f\|_2 = \left(\int_a^b w(x)\, |f(x)|^2 \, dx \right)^{\frac{1}{2}} \leq \left(\int_a^b w(x)\, (\max_{a \leq x \leq b} |f(x)|)^2 \, dx \right)^{\frac{1}{2}} = \|f\|_\infty \left(\int_a^b w(x)\, dx \right)^{\frac{1}{2}}.$$

2. $\forall \varepsilon > 0$ (however small) and $M > 0$ (however large) constants, $\exists\, f \in C[a,b]$ such that

   $$\|f\|_2 < \varepsilon, \qquad \|f\|_\infty > M.$$

   Solution: Consider the continuous function f on $[a,b]$ such that

   $$f(x) = \begin{cases} 0 & x \in [a, m-\delta) \\ (x - m + \delta)\frac{2M}{\delta} & x \in [m-\delta, m) \\ (m + \delta - x)\frac{2M}{\delta} & x \in [m, m+\delta) \\ 0 & x \in [m+\delta, b] \end{cases}$$

then $\|f\|_\infty > M$. Take δ such that $\frac{4}{\sqrt{3}} MW\sqrt{\delta} < \varepsilon$. Consider

$$\begin{aligned}
\|f\|_2 &= \left(\int_{m-\delta}^{m+\delta} w(x)\,|f(x)|^2\,dx \right)^{\frac{1}{2}} \\
&= \left(\frac{2M}{\delta} \int_{m-\delta}^{m} w(x)\,(x-m+\delta)^2\,dx + \frac{2M}{\delta} \int_{m}^{m+\delta} w(x)\,(m+\delta-x)^2\,dx \right)^{\frac{1}{2}} \\
&\le \frac{4M}{\delta} \left(\int_{m-\delta}^{m} w(x)\,dx \right)^{\frac{1}{2}} \left(\int_{m-\delta}^{m} (x-m+\delta)^2\,dx \right)^{\frac{1}{2}} \\
&= \frac{4MW}{\delta} \frac{\sqrt{\delta^3}}{\sqrt{3}} = \frac{4}{\sqrt{3}} MW\sqrt{\delta} < \varepsilon.
\end{aligned}$$

Problem 2

The Newton form of the interpolation polynomial on distinct nodes $x_0, \ldots, x_n$ is

$$p_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)\cdots(x-x_{n-1}). \quad (1)$$

The divided differences (DD) are defined as follows:

DD of order 1 $\quad f[x_0] := f(x_0)$

DD of order 2 $\quad f[x_0, x_1] := \frac{f[x_1]-f[x_0]}{x_1-x_0}$

$\vdots \qquad \vdots$

DD of order k $\quad f[x_0, x_1, \ldots, x_k] := \frac{f[x_1,\ldots,x_k]-f[x_0,\ldots,x_{k-1}]}{x_k-x_0}$

Prove the following results.

- **Theorem**: In the Newton form of the interpolation polynomial (1)

$$a_k := f[x_0, \ldots, x_k], \qquad k = 0, \ldots, n.$$

Guidelines: The proof is by induction.

  - For $k = 0$ $f(x_0) = p(x_0) = a_0$.
  - Assume $a_i := f[x_0, \ldots, x_i]$, $i = 0, \ldots, k-1$ are the coefficients of the Newton interpolation polynomial of degree k (induction

hypothesis). Prove then that $a_k := f[x_0, \dots, x_k]$ is the missing coefficient for the polynomial of degree $k+1$.

To prove this consider the Newton interpolation polynomial p_k of degree k (i.e. as in (1) with $n = k$) and evaluate it in x_k.

– Using the indusction hypothesis, you should be able to deduce that

$$f[x_0, x_k] = \frac{f(x_k) - a_0}{x_k - x_0},$$

then

$$f[x_0, x_1, x_k] = \frac{f[x_0, x_k] - a_1}{x_k - x_1},$$

and so repeating the procedure until

$$f[x_0, x_1, \dots, x_{k-1}, x_k] = \frac{f[x_0, \dots, x_{k-2}, x_k] - a_{k-1}}{x_k - x_{k-1}} = a_k,$$

which is what you want to prove. This will be enough to conclude the induction proof.

- **Theorem**: Let p_n be the interpolation polynomial on the distinct nodes $x_0, \dots, x_n$. Assume $t \neq x_i$, for $i = 0, \dots, n$, then

$$f(t) - p_n(t) = f[x_0, \dots, x_n, t] \prod_{j=0}^{n} (t - x_j).$$

Guidelines: It follows directly by considering the polynomial q interpolating f on the nodes $x_0, \dots, x_n, t$ in Newton form.

Solution: The polynomial q interpolating f on the nodes $x_0, \dots, x_n, t$ in Newton form is

$$q(x) = p(x) + f[x_0, \dots, x_n, t] \prod_{j=0}^{n} (x - x_j).$$

Since $q(t) = p(t)$, we get simply

$$f(t) - p(t) = f[x_0, \dots, x_n, t] \prod_{j=0}^{n} (x - x_j).$$

Problem 3

- Consider the following interpolation problem: find the polynomial $p \in \Pi_3$ satisfying

$$p_3(x_i) = f(x_i), \quad p_3'(x_i) = f'(x_i), \qquad i = 0, 1.$$

Here f is a continuous and differentiable function on the interval $[a, b]$ containing the nodes x_0 and x_1. Find p_3 assuming it has the form

$$p_3(x) = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^2(x - x_1). \qquad (2)$$

You need to determine the coefficients a, b, c, d using the values of f and its derivatives in x_0 and x_1.

- **Hermite interpolation on one node and with several derivatives.** Find the Hermite interpolation polynomial satisfying

$$p_n(x_0) = f(x_0), \quad p_n^{(k)}(x_0) = f^{(k)}(x_0)$$

for $k = 1, \ldots, n$. Here the superscript $^{(k)}$ denotes the k-derivative. What do you obtain?

Guidelines: modify the format (2) in an appropriate way so that it fits your problem. Then find the coefficients of the polynomial.

Solution: We consider

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n$$

then $a_0 = p_n(x_0) = f(x_0)$, $a_1 = p_n'(x_0) = f'(x_0)$ and $a_k = p_n^{(k)}(x_0) = f^{(k)}(x_0)$. So Hermite interpolation for this case reproduces the Taylor polynomial of degree n for f in x_0.