

# MA2501, Spring 2019, Numerical Methods

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## Problem 1

We denote by  $\sigma(A)$  the set of all eigenvalues of the square matrix  $A$ . Recall that the spectral radius of  $A$  is  $\rho(A) := \max_{\lambda \in \sigma(A)} |\lambda|$ . Recall that by definition the condition number of  $A$  with respect to the 2-norm is  $\mathcal{K}_2(A) := \|A\|_2 \|A^{-1}\|_2$ .

Prove that if  $A$  is a  $n \times n$  real, symmetric matrix then

$$\mathcal{K}_2(A) = \frac{\max_{\lambda \in \sigma(A)} |\lambda|}{\min_{\lambda \in \sigma(A)} |\lambda|}.$$

### Guidelines:

- Use in the proof the following definition of 2-norm,  $\|A\|_2 = \rho(A^T A)^{\frac{1}{2}}$  (see the book of Süli and Mayers, theorem 2.9 for this result).
- Prove that, for any matrix  $A$   $n \times n$ , if  $\lambda$  is an eigenvalue of  $A$  then  $\lambda^2$  is an eigenvalue of  $A^2$ .
- Prove that for  $A$  symmetric  $\|A\|_2 = \rho(A)$ .
- Prove that, for any matrix  $A$   $n \times n$ , if  $\lambda$  is an eigenvalue of  $A$  then  $\lambda^{-1}$  is an eigenvalue of  $A^{-1}$ .
- Prove that for  $A$  symmetric  $\|A^{-1}\|_2 = \frac{1}{\min_{\lambda \in \sigma(A)} |\lambda|}$ .

## Problem 2

### Householder transformations

An Householder transformation is a  $n \times n$  matrix of the form

$$P = I - 2ww^T,$$

where  $I$  is the identity matrix  $n \times n$ ,  $w \in \mathbb{R}^n$  with  $w$  of 2-norm equal to 1. Given  $x \in \mathbb{R}^n$ , we can define a Householder transformation such that

$$Px = \gamma \mathbf{e}_1, \quad \gamma \in \mathbb{R}$$

and  $\mathbf{e}_1$  the first canonical vector. If  $x = 0$ ,  $P\mathbf{0} = 0\mathbf{e}_1$ . If  $x \neq 0$ , this can be achieved by taking

$$w := \tilde{w}/\|\tilde{w}\|_2, \quad \text{where } \tilde{w} := x \pm \|x\|_2 \mathbf{e}_1. \quad (1)$$

- a) Prove that a Householder transformation is symmetric and orthogonal, that is  $P = P^T$  and  $P^T P = I$ , where  $I$  is the identity matrix.
- b) To compute the  $QR$  factorization of a matrix  $A$   $m \times n$ , one can use  $n$  Householder transformations. Consider the matrix

$$A := \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

apply two Householder transformations to transformations  $P_1$  and  $P_2$  to obtain an upper triangular  $2 \times 3$  matrix  $R$  such that

$$P_2 P_1 A = R$$

and

$$A = QR, \quad \text{where } Q := P_1^T P_2^T.$$

The last row of  $R$  is the zero row vector. Note  $P_2 = I - 2vv^T$  must be constructed such that the first row and column of  $P_1 A$  will remain unchanged when applying  $P_2$  to  $(P_1 A)$ , and only the last two elements of the second column of  $P_1 A$  are affected by the multiplication by  $P_2$ . This can be achieved taking the vector  $v \in \mathbb{R}^3$  with the first component equal to zero.

Show that  $Q$  is an orthogonal matrix, i.e.  $Q^T Q = Q Q^T = I$ .

**Solution:**

$$P_1 = \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix}, \quad P_1 A = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 1 \\ 0 & \sqrt{2} \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -b & -a \end{bmatrix} \quad P_2 P_1 A = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix},$$

with  $a = 1 - 2(1 - \sqrt{3})^2 / ((1 - \sqrt{3})^2 + 2)$ ,  $b = -2(1 - \sqrt{3})\sqrt{2} / ((1 - \sqrt{3})^2 + 2)$ .

$$Q = P_1^T P_2^T, \quad R = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}.$$

c) Consider the least squares problem

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

compute the normal equations and find the least squares solution. Use then the method based on the  $QR$ -factorization (Theorem 2.13) and verify that you obtain the same answer.

You will consider the  $QR$ -factorization from **b)** and to get  $\hat{Q}$  and  $\hat{R}$  as in theorem 2.13, you consider only the first two columns of  $Q$  and the first two rows and columns of  $R$ .

**Solution:**

Normal equations

$$\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Least squares solution  $x = 3/2 + 1/3 = 1.8333$ ,  $y = -1/3 = -0.3333$ .

To get  $\hat{Q}$  and  $\hat{R}$  from the theorem, we consider only the first two columns of  $Q$  and the first two rows and columns of  $R$ .

$$\hat{Q} = \begin{bmatrix} \sqrt{2}/2 & a \\ 0 & a \\ \sqrt{2}/2 & -a \end{bmatrix}, \quad \hat{R} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix},$$

then computing  $\hat{R}^{-1}\hat{Q}^T b$  one verifies that the obtained vector of  $\mathbb{R}^2$  coincides with the solution of the normal equations.

- d) Compute the SVD of  $A$ . The matrix  $A$  has rank equal to 2 (verify). Compute a rank one approximation of the matrix  $A$  using the SVD.

**Solution:**

$$V = \begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix} \quad \Sigma = \text{diag}(\sqrt{6}, 1), U = \begin{bmatrix} 0 & 0.9129 \\ 0.4472 & 0.3651 \\ -0.8944 & 0.1826 \end{bmatrix}$$

## Appendix

We outline here a procedure to compute by hand the SVD of a matrix  $A$   $m \times n$  (as stated in theorem 2.14). Numerical algorithms for the SVD are not covered in this course.

- Compute  $A^T A$ , this is a symmetric and positive definite  $n \times n$  matrix, compute the eigenvalues and a basis of orthonormal eigenvectors for  $A^T A$ .
- The square roots of the eigenvalues of  $A^T A$  are the singular values  $\sigma_i$ , which are also the diagonal elements of the matrix  $\Sigma$  of the SVD.
- The orthonormal eigenvectors  $v_1, \dots, v_n$  are the columns of the matrix  $V^T$   $n \times n$ .
- Consider next  $u_i := \frac{1}{\sigma_i} A v_i$ ,  $i = 1, \dots, n$ , these are the first  $n$  columns of the matrix  $U$   $m \times n$ . (Verify that these are orthonormal).