MA2501, Spring 2019, Numerical Methods

March 18th 2019

Local truncation error and global error

Local error

We want to find the local error σ_{n+1} of the trapezoidal rule method

$$y_{n+1} = y_n + \frac{1}{2}h(f(y_{n+1}) + f(y_n)),$$

for the numerical solution of the scalar initial value problem $y'(t) = f(y)$, with $y(0) = y_0$, and where $h = t_{n+1} - t_n$.

We use the following definition of the local truncation error

$$\sigma_{n+1} = y(t_{n+1}) - z_{n+1},$$

with z_{n+1} defined by

$$z_{n+1} = y(t_n) + \frac{1}{2}h(f(y(t_{n+1})) + f(y(t_n))),$$

and it is sufficient to investigate the case $n = 0$.

Explain how we obtain the following expression for σ_1

$$\sigma_1 := -\frac{1}{2}\int_0^h (h-x)\, x\, y'''(\xi(x))\, dx,$$

and using the mean value theorem for integrals or otherwise find

$$\sigma_1 = -\frac{1}{12}h^3 y'''(\tilde{\xi}),$$

for some $\tilde{\xi}$ in the interval $(0, h)$, where y is the solution of the initial value problem.

Guidelines

Write the exact solution at t_1 as

$$y(t_1) = y_0 + \int_0^h f(y(x))\, dx,$$

find a similar expression for z_1 using the trapezoid rule to approximate the integral. Using the error for Newton-Cotes formulae find the expression for σ_1.

For the second part of the exercise consider the mean value theorem for integrals.

Solution. For the exact solution we have

$$y(t_1) = y_0 + \int_0^h f(y(x))\, dx,$$

and z_1 can be interpreted as

$$z_1 = y_0 + \int_0^h g(x)\, dx$$

where $g(x)$ is the linear polynomial interpolating the values $(0, f(y_0))$ and $(h, f(y(t_1)))$. Recall that the error for such interpolation polynomial is

$$f(y(x)) - g(x) = \frac{1}{2!} \left.\frac{d^2 f(y(x))}{dx^2}\right|_{x=\tilde{x}} x\,(x-h) = \frac{1}{2!} y'''(\xi(x))\, x\,(x-h),$$

for $\tilde{x} \in (0, h)$ and where $\xi(x) = \tilde{x}$. This yields the first given expression for σ_1. Since $(h-x)\,x$ is nonnegative, by the mean value theorem for integrals there exists $\tilde{\xi} \in (0, h)$ such that

$$\sigma_1 = -\frac{1}{2} y'''(\tilde{\xi}) \int_0^h (h-x)\, x\, dx,$$

and the final result is obtained by computing the integral.

Global error estimate

Suppose f satisfies the Lipschitz condition

$$|f(t,u) - f(t,v)| \le L|u - v|,$$

for all real t, u, v where L is a positive constant independent of t, and that $|y'''(t)| \le M$ for some positive constant M independent of t. Show that the global error $e_n = y(t_n) - y_n$ satisfies the inequality

$$|e_{n+1}| \le \frac{h^3 M}{12} + (1 + \frac{1}{2}hL)|e_n| + \frac{1}{2}hL|e_{n+1}|.$$

Hint. Use that $e_{n+1} = y(t_{n+1}) - z_{n+1} + z_{n+1} - y_{n+1} = \sigma_{n+1} + z_{n+1} - y_{n+1}$.

Solution. Using the Lipschitz condition we observe that

$$|z_{n+1} - y_{n+1}| \le |e_n| + \frac{1}{2}hL|e_{n+1}| + \frac{1}{2}hL|e_n|,$$

which substituted in

$$|e_{n+1}| \le |\sigma_{n+1}| + |z_{n+1} - y_{n+1}|,$$

and together with $|\sigma_{n+1}| \le \frac{h^3 M}{12}$, gives the desired result.

Convergence

For a constant step-size $h > 0$ satisfying $hL < 2$, deduce that, if $y_0 = y(0)$, then

$$|e_n| \le \frac{h^2 M}{12L}\left[\left(\frac{1 + \frac{1}{2}hL}{1 - \frac{1}{2}hL}\right)^n - 1\right].$$

Solution. Since $1 - \frac{1}{2}hL > 0$, from the result of the previous exercise we obtain

$$|e_{n+1}| \le \frac{1}{1 - \frac{1}{2}hL}\left((1 + \frac{1}{2}hL)|e_n| + \frac{1}{12}h^3 M\right).$$

Let us define $Y := \frac{h^3 M}{12(1-\frac{1}{2}hL)}$ and $X := \frac{1+\frac{1}{2}hL}{1-\frac{1}{2}hL}$, so

$$|e_n| \le Y + X|e_{n-1}| \le Y + XY + X^2 Y + \cdots + X^{n-1}Y,$$

because $e_0 = 0$, and using the formula for the partial sums of the geometric series

$$|e_n| \le Y\frac{X^n - 1}{X - 1}.$$

Since $(X-1)^{-1} = \frac{1-\frac{1}{2}hL}{hL}$ one easily obtains the desired inequality.