Numerical Methods

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Summary from last time

- Bisection method and Newton method;
- Fixed point iterations and fixed point equation;
- Existence of fixed points: Brouwer's Theorem;
- Existence/Uniqueness of fixed points and convergence of fixed point iteration: Contraction mapping Theorem;

Bisection method and Newton method

Thm If $f : [a, b] \to \mathbb{R}$ is continuous in $[a, b]$ and

$$f(a)f(b) < 0, \qquad f(x) = 0 \quad \in \quad [a, b]$$

then there exists at east one zero of f in $[a, b]$.

1. Start $a_0 := a$ $b_0 := b$
2. Find $x_1 := \frac{a_0+b_0}{2}$.
3. Check:

$$\begin{array}{lll} f(a_0)f(x_1) < 0 & a_1 := a_0 & b_1 := x_1 \\ f(x_1)f(b_0) < 0 & a_1 := x_1, & b_1 := b_0 \end{array}$$

- Repeat from 2.

Newton method: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$. ($f$ differentiable)

Rewrite $f(x) = 0$ as $x = g(x)$, fixed point equation

Thm (Brouwer's Theorem: existence of fixed points)
If $g : [a, b] \to \mathbb{R}$ is continuous in $[a, b]$ ($[a, b]$ close and bounded) and

$$g([a, b]) \subset [a, b]$$

then there exists $\xi \in [a, b]$, s.t. $\xi = g(\xi)$.

Contractions

Def Let $g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, g is a contraction if there exists $0 < L < 1$ s. t.

$$|g(x) - g(y)| \leq L|x - y|, \qquad \forall x, y \in [a, b].$$

Thm 1.3 p.7 (Contraction mapping theorem: exsistence/uniqueness of fixed points, convergence of fixed point iteration)

Let $g : [a, b] \to \mathbb{R}$ be continuous on $[a, b] \subset \mathbb{R}$ (bounded and closed interval). Let $g([a, b]) \subset [a, b]$. Assume g is a contraction on $[a, b]$.

Then g has a unique fixed point $\xi \in [a, b]$, and the fixed point iteration $x_k = g(x_{k-1})$ converges to ξ for any $x_0 \in [a, b]$.