**Exercise set 5**

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**1** Assume that the *regula falsi* method (without modifications) is used for finding the solution of the equation $x^3 = 0$.

**a)** Show that during all the iterations one endpoint of the solution interval remains unchanged, unless the iteration finds the solution after the first step.

**b)** Denote by $c_k$ the result of the method at the $k$-th step. Show that

$$\lim_{k\to\infty} \frac{|c_{k+1}|}{|c_k|} = 1. \tag{1}$$

**Note:** It can be proven that the method of false position converges to the root, so you can assume $\lim_{k\to\infty} c_k = 0$.

**Remark:** A sequence $(c_k)_{k\in\mathbb{N}}$ converging to 0 and satisfying (1) is said to converge *sublinearly*. Sublinear converge of a sequence means that the number of iterations it takes to come closer to the limit by even one digit will become arbitrarily large. Thus numerical methods that have such sequences as output should usually be avoided.

**2** Apply fixed point iteration to the solution of the equation $\cos(x) = \frac{1}{2}\sin(x)$ using the mapping

$$\Phi(x) = x + \cos(x) - \frac{1}{2}\sin(x).$$

**a)** Show that the mapping $\Phi$ is a contraction on $[0, \pi/2]$ (note that you also have to show that $\Phi(x) \in [0, \pi/2]$ for all $x \in [0, \pi/2]$).

**b)** Compute the first five iterates of the fixed point iteration using the starting value $x^{(0)} = 0$.

**c)** Provide an estimate of the accuracy of the outcome of the method after the $5^{\text{th}}$ iteration. How many iterations will be needed to obtain a result with an error smaller than $10^{-12}$?

**3** Find an approximation of a solution of the equation $x^3 - 2x - 5 = 0$ by applying three steps of:

**a)** the bisection method with starting interval $[2, 3]$,

**b)** the secant method with starting values $x^{(0)} = 3$ and $x^{(1)} = 3.5$,

**c)** the Newton method with starting value $x^{(0)} = 3$.

**4** Compute the first three steps of the Newton method for the solution of the system of equations

$$-5x + 2\sin(x) + \cos(y) = 0,$$
$$4\cos(x) + 2\sin(y) - 5y = 0$$

with initial value $(x^{(0)}, y^{(0)}) = (0, 0)$ (you may want to use PYTHON for solving the linear systems).

**5** Implement in PYTHON Newton's method for the solution of an equation $f(x) = 0$ with $f\colon \mathbb{R} \to \mathbb{R}$. Your function should take as an input the function $f$, its derivative $f'$, and a starting value $x^{(0)}$. Test your implementation on the function in example **3** of this exercise set.

### PYTHON-comments

In PYTHON, you can create functions by defining them as lambda expressions. For example, $f(x) = x^2$ can be defined by `f = lambda x :  x**2`, and $g(x, y) = x + y$ can be defined as `f = lambda x,y :  x+y`. In this way, you can pass these function expressions as arguments in other functions inside your main PYTHON-script. Then you can evaluate directly by writing (f(x0)), where $x_0$ is the point to evaluate $f$ at.