Norwegian University of Science and Technology
Department of Mathematics

MA2501 Numerical Methods
Spring 2017

Solutions to exercise set 9

1 The procedure has been implemented in the MATLAB function `adsimpson.m` available on the course home page. The exact value of both integrals is $\pi \approx 3.14159....$

a) The procedure gives the approximation 3.14159 to the digits given. The actual absolute error is about 6.1×10^{-8}, well below ϵ.

b) The procedure gives the approximation 3.14159 to the digits given. The actual absolute error is about 1.8×10^{-8}, again well below ϵ.

2 The second Gauss–Legendre rule with $h = 1$ yields

$$\begin{aligned} Q(f, 0, 2, 1) = &\frac{5}{18} f\Big(\frac{1}{2} - \frac{1}{2\sqrt{3}}\Big) + \frac{8}{18} f\Big(\frac{1}{2}\Big) + \frac{5}{18} f\Big(\frac{1}{2} + \frac{1}{2\sqrt{3}}\Big) \\ &+ \frac{5}{18} f\Big(\frac{3}{2} - \frac{1}{2\sqrt{3}}\Big) + \frac{8}{18} f\Big(\frac{3}{2}\Big) + \frac{5}{18} f\Big(\frac{3}{2} + \frac{1}{2\sqrt{3}}\Big) \approx 0.4440. \end{aligned}$$

3

a) This result does not depend on the choice of x_0, x_1, x_2 as long as they are distinct. For any 3 distinct points the interpolation polynomial must equal f, when f is a polynomial of degree at most 2. This follows from the uniqueness theorem of interpolating polynomials. Since the rule is generated by integrating the polynomial precisely, it also integrates f precisely in this case.

b) Using the fact that 1, $x - \frac{1}{2}$ and $(x - \frac{1}{2})^2$ should be integrated exactly, gives the three equations:

$$\begin{aligned} 1 &= \int_0^1 1 dx = A_0 + A_1 + A_2 \\ 0 &= \int_0^1 x - \frac{1}{2} dx = \frac{1}{4}(-A_0 + A_2) \\ \frac{1}{12} &= \int_0^1 \Big(x - \frac{1}{2}\Big)^2 dx = \frac{1}{16}(A_0 + A_2) \end{aligned}$$

The second equation gives $A_0 = A_2$, from the third we then get $A_0 = A_2 = 2/3$ and finally from the first $A_1 = -1/3$.

c) From the definition we can immediately write down the Lagrange polynomials.

$$\ell_0 = \frac{\left(x-\frac{1}{2}\right)\left(x-\frac{3}{4}\right)}{\frac{1}{8}} = 8x^2 - 10x + 3$$
$$\ell_1 = \frac{\left(x-\frac{1}{4}\right)\left(x-\frac{3}{4}\right)}{-\frac{1}{16}} = -16x^2 + 16x - 3$$
$$\ell_0 = \frac{\left(x-\frac{1}{4}\right)\left(x-\frac{1}{2}\right)}{\frac{1}{8}} = 8x^2 - 6x + 1$$

which are trivially integrated to give

$$A_0 = \int_0^1 \ell_0 dx = \int_0^1 8x^2 - 10x + 3dx = 2/3$$
$$A_1 = \int_0^1 \ell_1 dx = \int_0^1 -16x^2 + 16x - 3dx = -1/3$$
$$A_2 = \int_0^1 \ell_1 dx = \int_0^1 8x^2 - 6x + 1dx = 2/3$$

so the resulting weights are the same. This must be the case. As discussed in a), since the functions 1, $x-\frac{1}{2}$ and $(x-\frac{1}{2})^2$ are all polynomials of degree less than or equal to 2 they match their interpolating polynomial through x_0, x_1, x_2. Consequently they will be integrated exactly when the weights are chosen as in c). However this was precisely the requirement we directly imposed in b), and since the resulting linear system had a unique solution, those weights must match those found in c).

Note also that 1, $x-\frac{1}{2}$ and $(x-\frac{1}{2})^2$ clearly form a basis for all polynomials of degree less than or equal to 2, i.e. any such polynomial can be written as a linear combination of these three functions. The linearity of integration and the quadrature rule then implies that the requirement in b) is equivalent to the requirement that all polynomials of degree less than or equal to 2 be integrated exactly.

d) We use the hint and check

$$0 = \int_0^1 \left(x-\frac{1}{2}\right)^3 dx = \frac{1}{64}\left(-\frac{2}{3}+\frac{2}{3}\right)$$
$$\frac{1}{80} = \int_0^1 \left(x-\frac{1}{2}\right)^4 dx \neq \frac{1}{256}\left(\frac{2}{3}+\frac{2}{3}\right) = \frac{1}{192}$$

Since 1, $x-\frac{1}{2}$, $(x-\frac{1}{2})^2$ and $(x-\frac{1}{2})^3$ clearly form a basis for all polynomials of degree less than or equal to 3. It again follows from linearity of integration and the quadrature rule that the formula is exact for all such polynomials. Because $(x-\frac{1}{2})^4$ is a polynomial of degree 4, the formula is clearly not exact for all polynomials of degree 4.

e) We use the hint and check

$$0 = \int_0^1 \left(x-\frac{1}{2}\right)^5 dx = \frac{1}{1024}\left(-\frac{2}{3}+\frac{2}{3}\right)$$

In fact it is readily observed that the formula will be exact for all $(x - \frac{1}{2})^{2n+1}$ with n some nonnegative integer, since it will be exact for all $f(x-1/2)$, where $f(x)$ is an odd function. This does not mean the formula is exact for general polynomials of degree 5. Suppose it were exact for one such polynomial $p(x)$. Then $p(x) + k(x - \frac{1}{2})^4$ for some constant $k \neq 0$ would also be a polynomial of degree 5. However, since the formula is not exact for $(x - \frac{1}{2})^4$ it follows from linearity that the quadrature rule will not be exact for this new polynomial.

f) A linear transformation gives

$$\int_a^b f(x)dx \approx I(f) = (b-a)\sum_{k=0}^{2} A_k f\left(a + (b-a)x_k\right)$$

with A_k and x_k as before.

4 Call the interval this quadrature formula applies to for $[a, b]$, where we exclude the trivial case $a = b$ and assume without loss of generality that $b > a$. Now, following the hint it is readily observed that

$$M^2(x) = (x-x_0)^2(x-x_1)^2 \cdots (x-x_n)^2,$$

is 0 at all the nodes $x_0, x_1, \ldots, x_n$. This implies that for a quadrature formula

$$I(f) = \sum_{k=0}^{n} A_k f(x_k),$$

$I(M^2) = 0$. However M^2 is obviously positive at all points except the nodes and continuous, and so the actual integral of M^2 must be positive, i.e.

$$\int_a^b M^2(x)\,dx > 0.$$

Thus M^2 will not be integrated exactly by the quadrature formula. The proof now follows from the realization that M^2 is a polynomial of degree $2n+2$.

5 Denote by S the interpolating linear spline with grid size $h = \pi/n$. Then we have the estimate

$$\max_{x\in[0,\pi]} |S(x) - f(x)| \leq \frac{h^2}{8} \max_{x\in[0,\pi]} |f''(x)|.$$

Now,

$$f''(x) = -10^4 \sin(100x),$$

and therefore

$$\max_{x\in[0,\pi]} |f''(x)| = 10^4.$$

Thus

$$\max_{x\in[0,\pi]} |S(x) - f(x)| \leq \frac{10^4}{8} h^2.$$

As a consequence, the error is guaranteed to be smaller than 10^{-8}, if

$$h^2 < 8 \times 10^{-12},$$

which, since $h = \pi/n$, is equivalent to the estimate

$$n > \frac{10^6 \pi}{\sqrt{8}}.$$