MA2501 Numerical Methods
Spring 2017

Solutions to exercise set 3

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1. Write a MATLAB program for the solution of a linear system $\mathbf{Ax} = \mathbf{b}$ in the case where $\mathbf{A}$ is tridiagonal. More precisely, the program should take the three non-zero diagonals of $\mathbf{A}$ and the vector $\mathbf{b}$ as input and use Gaussian elimination without pivoting for the solution of the system.

   Test your program on the matrix $\mathbf{A} \in \mathbb{R}^{200\times 200}$ with main diagonal $\mathbf{d} = [4, 4, \dots, 4]$ and lower and upper diagonals $\mathbf{a} = \mathbf{c} = [-1, -1, \dots, -1]$, and the right hand sides $\mathbf{b}_1 = [1, \dots, 1]^T$ and $\mathbf{b}_2 = [1, 2, 3, \dots, 200]^T$.

Possible solution:

Example code can be found on the webpage of the course.

2. **Cf. Cheney and Kincaid, Exercise 8.1.19**

   a) Prove that the product of two lower triangular matrices is lower triangular.

   b) Prove that the product of two unit lower triangular matrices is unit lower triangular.

   c) Prove that the inverse of an invertible lower triangular matrix is lower triangular.

   d) Prove that the inverse of a unit lower triangular matrix is unit lower triangular.

   e) Prove the previous statements for upper triangular matrices.

Possible solution:

a) Let $\mathbf{L}$, $\mathbf{M} \in \mathbb{R}^{n\times n}$ be two lower triangular matrices. Assume that $1 \le i < j \le n$. Since $\mathbf{L}$ and $\mathbf{M}$ are lower triangular we have $\ell_{ik} = 0$ for $k > i$ and $\ell_{kj} = 0$ for $k < j$. Thus

$$(\mathbf{LM})_{ij} = \sum_{k=1}^{n} \ell_{ik} m_{kj} = \sum_{k=1}^{i} \ell_{ik} \cdot 0 + \sum_{k=i+1}^{j-1} 0 \cdot 0 + \sum_{k=j}^{n} 0 \cdot m_{kj} = 0.$$

Thus $(\mathbf{LM})_{ij} = 0$ whenever $j > i$, proving that $\mathbf{LM}$ is lower diagonal.

b) Let $\mathbf{L}$, $\mathbf{M} \in \mathbb{R}^{n\times n}$ be unit lower triangular. From the previous part we already know that $\mathbf{LM}$ is lower triangular. Now let $1 \le i \le n$. Then

$$(\mathbf{LM})_{ii} = \sum_{k=1}^{n} \ell_{ik} m_{ki} = \sum_{k=1}^{i-1} \ell_{ik} \cdot 0 + \ell_{ii} m_{ii} + \sum_{k=i+1}^{n} 0 \cdot m_{ki} = \ell_{ii} m_{ii} = 1 \cdot 1 = 1.$$

Thus the diagonal entries of $\mathbf{LM}$ are all equal to 1, and therefore $\mathbf{LM}$ is unit lower triangular.

c) Let $\mathbf{L} \in \mathbb{R}^{n\times n}$ be an invertible lower triangular matrix with inverse $\mathbf{M} := L^{-1}$. Assume that $\mathbf{M}$ is not lower triangular. Then there exist indicies $1 \le i < j \le n$ such that $m_{ij} \neq 0$. Moreover, we can choose these indices in such a way that $m_{kj} = 0$ for all $k < i$. Since $\mathbf{LM}$ is the identity matrix, we have $(\mathbf{LM})_{ij} = 0$. Hence

$$0 = (\mathbf{LM})_{ij} = \sum_{k=1}^{n} \ell_{ik} m_{kj} = \sum_{k=1}^{i-1} \ell_{ik} \cdot 0 + \ell_{ii} m_{ij} + \sum_{k=i+1}^{n} 0 \cdot m_{kj} = \ell_{ii} m_{ij}.$$

Since $\mathbf{L}$ is an invertible lower triangular matrix its diagonal elements are different from 0. Thus the equation $0 = \ell_{ii} m_{ij}$ already implies that $m_{ij} = 0$, which is a contradiction to the definition of m_{ij}. Hence $\mathbf{M}$ is lower triangular.

d) Let $\mathbf{L}$ be unit lower triangular. Since $\det \mathbf{L} = 1$, it follows that $\mathbf{L}$ is invertible. Denote its inverse by $\mathbf{M} := \mathbf{L}^{-1}$. We have already shown that the matrix $\mathbf{M}$ is lower triangular. Hence we only have to prove that $\mathbf{M}_{ii} = 1$ for every i. Because $\mathbf{LM}$ is the identity matrix, we have, similar as in the second part of this exercise, the equation

$$1 = (\mathbf{LM})_{ii} = \sum_{k=1}^{n} \ell_{ik} m_{ik} = \ell_{ii} m_{ii}.$$

Since $\ell_{ii} = 1$, this proves that also $m_{ii} = 1$.

e) Assume that $\mathbf{U}$ and $\mathbf{V}$ are upper triangular. Then $\mathbf{U}^T$ and $\mathbf{V}^T$ are lower triangular. Moreover $(\mathbf{UV})^T = \mathbf{V}^T\mathbf{U}^T$. Thus, using the first part of this exercise we obtain that $(\mathbf{UV})^T$ is lower triangular and therefore $\mathbf{UV}$ is upper triangular. The proof for unit upper triangular matrices is similar. Finally, the fact that the inverse of a (unit) upper triangular matrix is (upper) triangular follows from the fact that $(\mathbf{U}^T)^{-1} = (\mathbf{U}^{-1})^T$; the former is lower triangular and therefore the latter as well.

3 **a)** Assume that $\mathbf{A} \in \mathbb{R}^{n\times n}$ is invertible and has an LU factorization. Prove that the LU factorization is unique.

b) Find matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$ that either do not have an LU factorization, or whose LU factorization is not unique.

Possible solution:

a) Assume that $\mathbf{A} = \mathbf{L}_1\mathbf{U}_1$ and $\mathbf{A} = \mathbf{L}_2\mathbf{U}_2$ are two LU factorizations of the matrix $\mathbf{A}$. Then

$$\mathbf{L}_1\mathbf{U}_1 = \mathbf{L}_2\mathbf{U}_2.$$

Since the matrix $\mathbf{A}$ is invertible, so are the matrices $\mathbf{L}_j$ and $\mathbf{U}_j$. Thus we can multiply the equation above with $\mathbf{L}_2^{-1}$ from the left and with $\mathbf{U}_1^{-1}$ from the right. Thus we obtain the equation

$$\mathbf{L}_2^{-1}\mathbf{L}_1 = \mathbf{U}_2\mathbf{U}_1^{-1}.$$

Since $\mathbf{L}_2$ and $\mathbf{L}_1$ are unit lower triangular, so is $\mathbf{L}_2^{-1}\mathbf{L}_1$. Since $\mathbf{U}_2$ and $\mathbf{U}_1$ are upper triangular, so is $\mathbf{U}_2\mathbf{U}_1^{-1}$. Thus $\mathbf{L}_2^{-1}\mathbf{L}_1$ is a unit lower triangular matrix that is at the same time upper triangular. The only possibility is therefore that $\mathbf{L}_2^{-1}\mathbf{L}_1$ is the identity matrix. This shows that we have $\mathbf{L}_2 = \mathbf{L}_1$ and, similarly, $\mathbf{U}_2 = \mathbf{U}_1$. Thus the LU factorization of $\mathbf{A}$ is unique.

b) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Assume that $\mathbf{A} = \mathbf{LU}$ is an LU factorization of $\mathbf{A}$. Since $\ell_{11} = 1$ and $\ell_{12} = 0$, it follows that $u_{11} = a_{11} = 0$. This implies, however, that the matrix $\mathbf{U}$ is not invertible. As a consequence, also the product $\mathbf{LU}$ is not invertible. This contradicts, however, the fact that $\det \mathbf{A} = -1$, which would imply that $\mathbf{A} = \mathbf{LU}$ is an invertible matrix. Thus, $\mathbf{A}$ cannot have an LU factorization.

Consider now the case where $n \geq 2$ and $\mathbf{A} \in \mathbb{R}^{n\times n}$ is the 0-matrix. Let $\mathbf{L}$ be *any* unit lower triangular matrix and let $\mathbf{U} = \mathbf{0} \in \mathbb{R}^{n\times n}$. Then $\mathbf{LU} = \mathbf{0} = \mathbf{A}$, implying that we have constructed an LU factorization of $\mathbf{A}$. Since there is more than one unit lower triangular matrix in $\mathbb{R}^{n\times n}$ with $n \geq 2$, this proves the non-uniqueness of the factorization.

4 Factor the following matrices into the LU decomposition using the LU Factorization Algorithm where $l_{ii} = 1$ for all i.

a)

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1.012 & -2.132 & 3.104 \\ -2.132 & 4.906 & -7.013 \\ 3.104 & -7.013 & 0.014 \end{bmatrix}$$

c)

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1.5 & 0 & 0 \\ 0 & -3 & 0.5 & 0 \\ 2 & -2 & 1 & 1 \end{bmatrix}$$

d)

$$\begin{bmatrix} 2.1756 & 4.0231 & -2.1732 & 5.1967 \\ -4.0231 & 6.0000 & 0 & 1.1973 \\ -1.000 & -5.2107 & 1.1111 & 0 \\ 6.0235 & 7.0000 & 0 & -4.1561 \end{bmatrix}$$

Possible solution:

a)

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1.5 & 1 & 0 \\ 1.5 & 1 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 0 & 4 \end{bmatrix}.$$

b)

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -2.106719 & 1 & 0 \\ 3.067194 & -1.142997 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 1.012 & -2.132 & 3.104 \\ 0 & 0.4144743 & -0.4737431 \\ 0 & 0 & -10.04806 \end{bmatrix}.$$

c)

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 1 & -1.33333 & 2 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

d)

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1.849191 & 1 & 0 & 0 \\ -0.4596433 & -0.2501219 & 1 & 0 \\ 2.768662 & -0.3079436 & -5.352283 & 1 \end{bmatrix},$$

$$\mathbf{U} = \begin{bmatrix} 2.1756 & 4.0231 & -2.1732 & 5.1967 \\ 0 & 13.43948 & -4.018662 & 10.80699 \\ 0 & 0 & -0.8929524 & 5.091694 \\ 0 & 0 & 0 & 12.03613 \end{bmatrix}.$$