Norwegian University of Science
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MA2501 Numerical Methods
Spring 2016

Solutions to exercise set 11

1 Consider the initial value problem

$$\begin{aligned} x' &= \sin(t^2 + x), \\ x(0) &= 0. \end{aligned}$$

a) Use Euler's method with a step size of $h = 1/4$ in order to obtain an approximation of $x(2)$.

b) Use the improved Euler method (Heun's second order method) with a step size of $h = 1/2$ in order to obtain an approximation of $x(2)$.

c) Use the classical Runge–Kutta method with a step size of $h = 1$ in order to obtain an approximation of $x(2)$.

Possible solution:

a) Euler's method for this particular initial value problem takes the form

$$x_{k+1} = x_k + h \sin(t_k^2 + x_k),$$

with initial values

$$x_0 = x(t_0) = 0 \qquad \text{and} \qquad t_0 = 0.$$

Here

$$t_k = t_0 + kh, \quad k = 0, 1, 2, \ldots$$

With a step size of $h = 1/4$, we therefore obtain the (rounded) iterates

$$\begin{aligned} x_1 &= 0 + \frac{1}{4}\sin(0) = 0, \\ x_2 &= 0 + \frac{1}{4}\sin(1/16) = 0.015615, \\ x_3 &= 0.015615 + \frac{1}{4}\sin(1/4 + 0.015615) = 0.081240, \\ x_4 &= 0.081240 + \frac{1}{4}\sin(9/16 + 0.081240) = 0.231288, \\ x_5 &= 0.231288 + \frac{1}{4}\sin(1 + 0.231288) = 0.467018, \\ x_6 &= 0.467018 + \frac{1}{4}\sin(25/16 + 0.467018) = 0.691173, \\ x_7 &= 0.691173 + \frac{1}{4}\sin(9/4 + 0.691173) = 0.740943, \\ x_8 &= 0.740943 + \frac{1}{4}\sin(49/16 + 0.740943) = 0.587299, \end{aligned}$$

The value $x_8 = 0.587299$ is the desired approximation of $x(2)$.

b) The improved Euler method for this problem takes the form

$$\hat{x}_{k+1} = x_k + h\sin(t_k^2 + x_k),$$
$$x_{k+1} = x_k + \frac{h}{2}\big(\sin(t_k^2 + x_k) + \sin((t_k + h)^2 + \hat{x}_{k+1})\big).$$

With a step size of $h = 1/2$, we obtain the (rounded) iterates

$$\hat{x}_1 = 0 + \frac{1}{2}\sin(0 + 0) = 0,$$
$$x_1 = 0 + \frac{1}{4}\big(\sin(0 + 0) + \sin(1/4 + 0)\big) = 0.061851,$$
$$\hat{x}_2 = 0.061851 + \frac{1}{2}\sin(1/4 + 0.061851) = 0.215261,$$
$$x_2 = 0.061851 + \frac{1}{4}\big(\sin(1/4 + 0.061851) + \sin(1 + 0.215261)\big) = 0.372921,$$
$$\hat{x}_3 = 0.372921 + \frac{1}{2}\sin(1 + 0.372921) = 0.863165,$$
$$x_3 = 0.372921 + \frac{1}{4}\big(\sin(1 + 0.372921) + \sin(9/4 + 0.863165)\big) = 0.625149,$$
$$\hat{x}_4 = 0.625149 + \frac{1}{2}\sin(9/4 + 0.625149) = 0.756800,$$
$$x_4 = 0.756800 + \frac{1}{4}\big(\sin(9/4 + 0.625149) + \sin(4 + 0.756800)\big) = 0.441221.$$

The value $x_4 = 0.441221$ is the desired approximation of $x(2)$.

c) The classical Runge–Kutta method uses the iteration:

$$K_1 = hf(t_k, x_k),$$
$$K_2 = hf(t_k + \frac{h}{2}, x_k + \frac{1}{2}K_1),$$
$$K_3 = hf(t_k + \frac{h}{2}, x_k + \frac{1}{2}K_2),$$
$$K_4 = hf(t_k + h, x_k + K_3),$$

and

$$x_{k+1} = x_k + \frac{1}{6}\Big(K_1 + 2K_2 + 2K_3 + K_4\Big).$$

In this particular example, we obtain with a step size of $h = 1$ the following iterates:

- First step:

$$K_1 = 0,$$
$$K_2 = 0.247404,$$
$$K_3 = 0.365064,$$
$$K_4 = 0.978912,$$

and

$$x_1 = 0.367308.$$

- Second step:

$$K_1 = 0.979368,$$
$$K_2 = 0.034594,$$
$$K_3 = 0.485546,$$
$$K_4 = -0.990151,$$

and

$$x_2 = 0.538891,$$

which is also the desired approximation of $x(2)$.

2 Assume that the function $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and decreasing in its second component. That is,

$$f(t,x) \leq f(t,z) \qquad \text{whenever} \qquad x \geq z.$$

Show that the implicit Euler method for the solution of the differential equation

$$x' = f(t,x),$$
$$x(t_0) = x_0,$$

is well-defined. That is, regardless of the step-size $h > 0$, the (non-linear) equation that has to be solved in each iteration has a unique solution.

Possible solution:

In the k-th step of the implicit Euler method, we have to solve the equation

$$x_{k+1} = x_k + hf(t_k + h, x_{k+1}).$$

Defining

$$g_k(z) := x_k + hf(t_k + h, z) - z,$$

this means that x_{k+1} solves the equation

$$g_k(x_{k+1}) = 0.$$

By assumption, the function g_k is continuous and strictly decreasing (because $f(t_k + h, \cdot)$ is decreasing and $-z$ is strictly decreasing). Moreover, for $z > 0$ we have

$$g_k(z) = x_k + hf(t_k + h, z) - z \leq x_k + hf(t_k + h, 0) - z = g_k(0) - z,$$

showing that $g_k(z) \to -\infty$ for $z \to +\infty$. Similarly, for $z < 0$ we have

$$g_k(z) \geq g_k(0) - z \to_{z \to -\infty} +\infty.$$

Since g_k is continuous and strictly decreasing, it follows that the equation $g_k(z) = 0$ has a unique solution, which is what we wanted to show.

3 Compute three steps with step size $h = 1$ for the numerical solution of the differential equation

$$x' = -3x - e^x,$$
$$x(0) = 1,$$

using:

a) Euler's method.

b) The improved Euler method.

c) The implicit Euler method. Numerically solve the non-linear equations you obtain by performing two steps of Newton's method in each step of the implicit Euler method.

Possible solution:

a) Here the iteration (with $h = 1$) is

$$x_{k+1} = x_k + (-3x_k - e^{x_k}) = -2x_k - e^{x_k}.$$

Thus we obtain

$$\begin{aligned} x_0 &= 1, \\ x_1 &= -2 - e \approx -4.718282, \\ x_2 &\approx 9.427633, \\ x_3 &\approx -12446. \end{aligned}$$

b) Here the iteration is

$$\begin{aligned} \hat{x}_{k+1} &= x_k + (-3x_k - e^{x_k}), \\ x_{k+1} &= x_k + \frac{1}{2}\big(-3x_k - e^{x_k} - 3\hat{x}_{k+1} - e^{\hat{x}_{k+1}}\big). \end{aligned}$$

We obtain

$$\begin{aligned} \hat{x}_1 &\approx -4.718282, & x_1 &\approx 5.213817, \\ \hat{x}_2 &\approx -194.222, & x_2 &\approx 196.829, \\ \hat{x}_3 &\approx -3 \cdot 10^{85}, & x_3 &\approx 3 \cdot 10^{85}. \end{aligned}$$

c) The implicit Euler method requires the solution of the equation

$$x_{k+1} = x_k - 3x_{k+1} - e^{x_{k+1}}$$

with respect to x_{k+1} in each step. Setting

$$g(z) := 4z + e^z,$$

this means that x_{k+1} solves the equation

$$g(x_{k+1}) - x_k = 0.$$

We solve this equation using two steps of Newton's method with a suitable initial guess, for instance the result with the explicit Euler method. We then have (for each k) the iteration

$$x_{k+1,0} := x_k - 3x_k - e^{x_k}$$

(this is the explicit Euler step) and

$$x_{k+1,\ell+1} := x_{k+1,\ell} - \frac{g(x_{k+1,\ell}) - x_k}{g'(x_{k+1,\ell})}$$

(this is a Newton step), or, using the definition of g,

$$x_{k+1,\ell+1} := x_{k+1,\ell} - \frac{4x_{k+1,\ell} + e^{x_{k+1,\ell}} - x_k}{4 + e^{x_{k+1,\ell}}}.$$

In this exercise, we stop the Newton iteration after two steps, that is, we set

$$x_{k+1} := x_{k+1,2}.$$

We therefore obtain:

- First Euler step:

  $$x_{1,0} := -2 - e \approx -4.718282,$$

  and then

  $$x_{1,1} = x_{1,0} - \frac{4x_{1,0} + e^{x_{1,0}} - 1}{4 + e^{x_{1,0}}} \approx 0.23670,$$
  $$x_{1,2} = x_{1,1} - \frac{4x_{1,1} + e^{x_{1,1}} - 1}{4 + e^{x_{1,1}}} \approx 0.00624.$$

  Thus

  $$x_1 = x_{1,2} \approx 0.00624.$$

- Second Euler step:

  $$x_{2,0} := -2x_1 - e^{x_1} \approx -1.01873,$$

  and then

  $$x_{2,1} = x_{2,0} - \frac{4x_{2,0} + e^{x_{2,0}} - 0.00624}{4 + e^{x_{2,0}}} \approx -0.16570,$$
  $$x_{2,2} = x_{2,1} - \frac{4x_{2,1} + e^{x_{2,1}} - 0.00624}{4 + e^{x_{2,1}}} \approx -0.20248.$$

  Thus

  $$x_2 = x_{2,2} \approx -0.20248.$$

- Third Euler step:

  $$x_{3,0} := -2x_2 - e^{x_2} \approx -0.41176,$$

  and then

  $$x_{3,1} = x_{3,0} - \frac{4x_{3,0} + e^{x_{3,0}} + 0.20248}{4 + e^{x_{3,0}}} \approx -0.24402,$$
  $$x_{3,2} = x_{3,1} - \frac{4x_{3,1} + e^{x_{3,1}} + 0.20248}{4 + e^{x_{3,1}}} \approx -0.24608.$$

  Thus

  $$x_3 = x_{3,2} \approx -0.24608.$$

Some additional remarks are here in order: Writing the differential equation in this exercise as $x' = f(x)$ with $f(x) = -3x - e^x$, we see that the function f is strictly decreasing with a unique zero $\hat{x} \in \mathbb{R}$ (with $\hat{x} \approx -0.25763$). In this case it is easy to show that every solution of the differential equation (regardless of the initial value) has the property that $x(t) \to \hat{x}$ for $t \to \infty$. Moreover this convergence is monotoneous: if $x(0) > \hat{x}$, then $x(t)$ is strictly decreasing to $\hat{x}$; if $x(0) < \hat{x}$, then $x(t)$ is strictly increasing to $\hat{x}$. If we compare this expected theoretical behaviour with the actual behaviour of the numerical solution, we see that:

- The solution with the Euler method is not monotoneous but instead jumps in each step over $\hat{x}$. Moreover, after three iterations it is quite far from the actual solution; after five iterations it is numerically equal to $-\infty$.
- The solution with the improved Euler method is increasing instead of decreasing, and after three iterations the result is (obviously) way off.
- The result of the implicit Euler method (using only two Newton steps) matches exactly the expected behaviour. The iterates are monotoneous and appear to be converging to the correct value. The added work of having to solve a non-linear system at each step, since the method is implicit, appears to be well worth it in this case.

4 The third order Adams–Bashforth method has the form

$$\begin{aligned} x_{k+1} &= x_k + h\Big(\frac{23}{12}f_k - \frac{16}{12}f_{k-1} + \frac{5}{12}f_{k-2}\Big), \\ t_{k+1} &= t_k + h, \\ f_{k+1} &= f(t_{k+1}, x_{k+1}). \end{aligned}$$

Show that this method can be derived by interpolating the function $\tau \mapsto f(\tau, x(\tau))$ in the points t_{k-2}, t_{k-1}, and t_k and then integrating the resulting quadratic polynomial.

Possible solution:

The Newton form of the interpolation polynomial through the points (t_k, f_k), (t_{k-1}, f_{k-1}), and (t_{k-2}, f_{k-2}) (which can be computed using divided differences) is

$$p(\tau) = f_k + (\tau - t_k)\frac{f_k - f_{k-1}}{h} + (\tau - t_k)(\tau - t_{k-1})\frac{f_k - 2f_{k-1} + f_{k-2}}{2h^2}.$$

Thus

$$\begin{aligned} &\int_{t_k}^{t_k+h} p(\tau)\,d\tau \\ &= f_k h + \frac{f_k - f_{k-1}}{h}\int_{t_k}^{t_k+h}(\tau - t_k)\,d\tau + \frac{f_k - 2f_{k-1} + f_{k-2}}{2h^2}\int_{t_k}^{t_k+h}(\tau - t_k)(\tau - t_{k-1})\,d\tau \\ &= f_k h + \frac{f_k - f_{k-1}}{h}\frac{h^2}{2} + \frac{f_k - 2f_{k-1} + f_{k-2}}{2h^2}\frac{5h^3}{6}. \end{aligned}$$

Therefore,

$$\begin{aligned} x_{k+1} &:= x_k + \int_{t_k}^{t_k+h} p(\tau)\,d\tau \\ &= x_k + f_k h + \frac{f_k - f_{k-1}}{2}h + \frac{5f_k - 10f_{k-1} + 5f_{k-2}}{12}h \\ &= x_k + \frac{23}{12}f_k h - \frac{16}{12}f_{k-1}h + \frac{5}{12}f_{k-2}h. \end{aligned}$$

5 Consider the differential equation

$$\begin{aligned} x' &= x - \frac{x^2}{2}, \\ x(0) &= 1. \end{aligned}$$

Compute three steps with step size $h = 1$ using:

a) The second order Adams–Bashforth method, which for an autonomous ODE is defined by,

$$x_{k+1} = x_k + h\Big(\frac{3}{2}f_k - \frac{1}{2}f_{k-1}\Big),$$

where $f_k \equiv f(x_k)$.

b) The second order Adams–Moulton method, which for an autonomous ODE is (implicitly) defined by

$$x_{k+1} = x_k + h\Big(\frac{1}{2}f_{k+1} + \frac{1}{2}f_k\Big).$$

c) The second order Adams–Bashforth–Moulton method.

In all three cases, use the improved Euler method for computing the iterate x_1.

Possible solution:

We first apply the improved Euler method in order to computing the first iterate x_1. The general formula for this method is (for an autonomous ODE)

$$x_{k+1} = x_k + \frac{h}{2}\big(f(x_k) + f(x_k + hf(x_k))\big).$$

With $h = 1$ and $x_0 = 1$ we obtain first

$$f(x_0 + f(x_0)) = f(1 + (1 - 1/2)) = f(3/2) = \frac{3}{2} - \frac{9}{8} = \frac{3}{8}$$

and then

$$x_1 = 1 + \frac{1}{2}\Big(\frac{1}{2} + \frac{3}{8}\Big) = \frac{23}{16} = 1.4375.$$

As a consequence,

$$f_1 = f(x_1) \approx 0.4043.$$

a) For the Adams–Bashforth method of second order

$$x_2 = x_1 + \frac{3}{2}f_1 - \frac{1}{2}f_0 = 1.4375 + \frac{3}{2}0.4043 - \frac{1}{2}\frac{1}{2} = 1.7939$$

and

$$f_2 = f(1.7939) = 0.1848;$$

then

$$x_3 = x_2 + \frac{3}{2}f_2 - \frac{1}{2}f_1 = 1.7939 + \frac{3}{2}0.1848 - \frac{1}{2}0.4043 = 1.8690$$

and

$$f_2 = f(1.8690) = 0.1224;$$

finally,

$$x_4 = x_3 + \frac{3}{2}f_3 - \frac{1}{2}f_2 = 1.8690 + \frac{3}{2}0.1848 - \frac{1}{2}0.1224 = 1.9602$$

and

$$f_4 = f(1.9602) = 0.0390.$$

b) For the Adams–Moulton method of second order we have for this particular function with step size $h = 1$

$$x_{k+1} = x_k + \frac{1}{2}f_k + \frac{1}{2}\Big(x_{k+1} - \frac{x_{k+1}^2}{2}\Big)$$

or

$$\frac{1}{4}x_{k+1}^2 + \frac{1}{2}x_{k+1} - x_k - \frac{1}{2}f_k = 0,$$

which means that

$$x_{k+1} = -1 \pm \sqrt{1 + 4x_k + 2f_k}.$$

Because $x_0 = 1 > -1$, it makes sense to choose the positive root[1] in this expression and thus set

$$x_{k+1} = -1 + \sqrt{1 + 4x_k + 2f_k}.$$

We obtain

$$\begin{aligned} x_2 &\approx 1.7493, \qquad f_2 \approx 0.2193, \\ x_3 &\approx 1.9044, \qquad f_3 \approx 0.0910, \\ x_4 &\approx 1.9664, \qquad f_4 \approx 0.0330. \end{aligned}$$

(Note that here it would have been possible (and sensible) to compute also x_1 with the Adams–Moulton methd.)

c) The Adams–Bashforth–Moulton method of second order is defined by the iteration

$$\begin{aligned} \hat{x}_{k+1} &= x_k + h\Big(\frac{3}{2}f_k - \frac{1}{2}f_{k-1}\Big), \\ \hat{f}_{k+1} &= f(t_{k+1}, \hat{x}_{k+1}), \\ x_{k+1} &= x_k + h\Big(\frac{1}{2}\hat{f}_{k+1} + \frac{1}{2}f_k\Big). \end{aligned}$$

[1] More precisely, it is easy to see that the differential equation has the two constant solutions $x = 0$ and $x = 2$. The solution of an initial value problem with $0 < x(0) < 2$ therefore satisfies $0 < x(t) < 2$ for all $t > 0$. In addition, because $f(x) > 0$ for $0 < x < 2$, it follows that in this case the solution is a strictly increasing function. Now note that the iteration can be explicitly written as $x_{k+1} = -1 + \sqrt{1 + 6x_k - x_k^2}$. One can show that this iteration has the same properties (that is, it is increasing and remains smaller than 2) for all starting values between 0 and 2.

Here we obtain in the first step

$$\begin{aligned}
\hat{x}_2 &= x_1 + \frac{3}{2}f_1 - \frac{1}{2}f_0 \approx 1.7939,\\
\hat{f}_2 &= f(\hat{x}_2) \approx 0.1848,\\
x_2 &= x_1 + \frac{1}{2}f_1 + \frac{1}{2}\hat{f}_2 \approx 1.7321,\\
f_2 &= f(x_2) \approx 0.2320,
\end{aligned}$$

then

$$\begin{aligned}
\hat{x}_3 &= x_2 + \frac{3}{2}f_2 - \frac{1}{2}f_1 \approx 1.8780,\\
\hat{f}_3 &= f(\hat{x}_3) \approx 0.1146,\\
x_3 &= x_2 + \frac{1}{2}f_2 + \frac{1}{2}\hat{f}_3 \approx 1.9054,\\
f_3 &= f(x_3) \approx 0.0902,
\end{aligned}$$

and finally

$$\begin{aligned}
\hat{x}_4 &= x_3 + \frac{3}{2}f_3 - \frac{1}{2}f_2 \approx 1.9246,\\
\hat{f}_4 &= f(\hat{x}_3) \approx 0.0726,\\
x_4 &= x_3 + \frac{1}{2}f_3 + \frac{1}{2}\hat{f}_4 \approx 1.9867,\\
f_4 &= f(x_4) \approx 0.0132.
\end{aligned}$$

6 Consider the second order initial value problem

$$\begin{aligned}
x'' &= -\sin(x) + x',\\
x(0) &= 0,\\
x'(0) &= 1.
\end{aligned}$$

a) Rewrite the second order equation as a system of first order equations.

b) Compute two steps of the classical Runge–Kutte method with step size $h = 1/2$ in order to obtain an approximation of $x(1)$.

Possible solution:

a) The equation can be rewritten as the system of first order equations

$$\begin{aligned}
x_1' &= x_2, & x_1(0) &= 0,\\
x_2' &= -\sin(x_1) + x_2, & x_2(0) &= 1.
\end{aligned}$$

b) To avoid double subscripts we here denote by $x_i^{(k)}$ the approximation from the method to the component function $x_i(t)$ at time $t_k = 0 + kh$ and let $\mathbf{X}_k = [x_1^{(k)}, x_2^{(k)}]^T$. The classical Runge–Kutta method with step size $h = 1/2$ yields:

- First step:

$$\mathbf{K}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$\mathbf{K}_2 = \begin{bmatrix} 1.25 \\ 1.0026 \end{bmatrix},$$
$$\mathbf{K}_3 = \begin{bmatrix} 1.2506 \\ 0.9432 \end{bmatrix},$$
$$\mathbf{K}_4 = \begin{bmatrix} 1.4716 \\ 0.8862 \end{bmatrix},$$

  and

$$\mathbf{X}_1 = \begin{bmatrix} 0.6227 \\ 14815 \end{bmatrix}.$$

- Second step:

$$\mathbf{K}_1 = \begin{bmatrix} 1.4815 \\ 0.8982 \end{bmatrix},$$
$$\mathbf{K}_2 = \begin{bmatrix} 1.7060 \\ 0.8683 \end{bmatrix},$$
$$\mathbf{K}_3 = \begin{bmatrix} 1.6986 \\ 0.8315 \end{bmatrix},$$
$$\mathbf{K}_4 = \begin{bmatrix} 1.8972 \\ 0.9021 \end{bmatrix},$$

  and

$$\mathbf{X}_2 = \begin{bmatrix} 1.4717 \\ 1.9148 \end{bmatrix}.$$

The first entry of $\mathbf{X}_2$ is an approximation of $x(1)$, that is, $x(1) \approx 1.4717$.

7 **a)** Construct the free cubic spline for the following data.

x	$f(x)$
0.1	−0.62049958
0.2	−0.28398668
0.3	0.00660095
0.4	0.24842440

b) The data in the previous task were constructed using the function

$$f(x) = x \cos x - 2x^2 + 3x - 1$$

Use the cubic spline to approximate $f(0.25))$ and $f'(0.25)$ and calculate the actual error in both cases.

Possible solution:

a) We follow the algorithm to find z_1 and z_2. All the $h_i = 0.1$ and consequently all $u_i = 0.4$. From $b_i = (y_{i+1} - y_i)/h_i$ and $v_i = 6(b_i - b_{i-1})$ we easily compute $\mathbf{b} =$

$[b_0, b_1, b_2]^T = [3.36512902.90587632.4182345]^T$ and $\mathbf{v} = [v_1, v_2]^T = [-2.7555162, -2.9258508]^T$. This gives the tridiagonal linear system

$$\begin{bmatrix} 0.4 & 0.1 \\ 0.1 & 0.4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -2.7555162 \\ -2.9258508 \end{bmatrix},$$

which is easily solved for $z_1 = -5.397476$ and $z_2 = -5.965258$. Insertion of this and the other known quantitites into the cubic equation formula for $S_i(x)$ gives.

$$\begin{aligned}
S_0(x) &= \frac{-5.397476}{6 \cdot 0.1}(x - 0.1)^3 + \left(\frac{-0.28398668}{0.1} - \frac{0.1}{6}(-5.397476)\right)(x - 0.1) \\
&\quad + \left(\frac{-0.62049958}{0.1}\right)(0.2 - x) \\
S_1(x) &= \frac{-5.965258}{6 \cdot 0.1}(x - 0.2)^3 + \frac{-5.397476}{6 \cdot 0.1}(0.3 - x)^3 \\
&\quad + \left(\frac{0.00660095}{0.1} - \frac{0.1}{6}(-5.965258)\right)(x - 0.2) \\
&\quad + \left(\frac{-0.28398668}{0.1} - \frac{0.1}{6}(-5.397476)\right)(0.3 - x) \\
S_2(x) &= \frac{-5.965258}{6 \cdot 0.1}(0.4 - x)^3 + \left(\frac{0.24842440}{0.1}\right)(x - 0.3) \\
&\quad + \left(\frac{0.00660095}{0.1} - \frac{0.1}{6}(-5.965258)\right)(0.4 - x)
\end{aligned}$$

or when simplified

$$S(x) = \begin{cases} S_0(x) = -8.9957933x^3 + 2.6987380x^2 + 3.1852131x - 0.95701248, & x \in [0.1, 0.2], \\ S_1(x) = -0.94630333x^3 - 2.1309560x^2 + 4.1511519x - 1.0214084, & x \in [0.2, 0.3], \\ S_2(x) = 9.9420966x^3 - 11.930516x^2 + 7.0910199x - 1.3153952, & x \in [0.3, 0.4]. \end{cases}$$

b) We begin with the function itself.

$$\begin{aligned}
S(0.25) = S_1(0.25) &= -0.94630333 \cdot 0.25^3 - 2.1309560 \cdot 0.25^2 + 4.1511519 \cdot 0.25 \\
&\quad - 1.0214084 = -0.13159116, \\
f(0.25) &= 0.25 \cos 0.25 - 2 \cdot 0.25^2 + 3 \cdot 0.25 - 1 = -0.13277189, \\
&|S(0.25 - f(0.25)| = 1.1807 \times 10^{-3}.
\end{aligned}$$

It is trivial to compute $f'(x) = \cos x - x \sin x - 4x + 3$, and to differentiate the spline function $S_1(x)$, so for the derivative

$$\begin{aligned}
S'(0.25) = S_1'(0.25) &= -3 \cdot 0.94630333 \cdot 0.25^2 - 2 \cdot 2.1309560 \cdot 0.25 + 4.1511519 \\
&= 2.9082421 \\
f'(0.25) &= \cos 0.25 - 0.25 \sin 0.25 - 4 \cdot 0.25 + 3 = 2.9070614, \\
&|S'(0.25 - f'(0.25)| = 1.1806 \times 10^{-3}.
\end{aligned}$$

We see that the accuracy of the approximation is very similar in both cases. Note that this is not generally the case.