MA2501 Numerical Methods
Spring 2016

Solutions to exercise set 10

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1 Compute a numerical approximation of the definite integral

$$\int_0^2 e^{-x^2} \sin(x)\, dx,$$

using the composite Gauss–Legendre rule with $n = 2$ and two subintervals, i.e. $h = 1$. (You may want to use MATLAB for the computations.)

Possible solution:

The second Gauss–Legendre rule with $h = 1$ yields

$$Q(f, 0, 2, 1) = \frac{5}{18} f\Big(\frac{1}{2} - \frac{\sqrt{3}}{2\sqrt{5}}\Big) + \frac{8}{18} f\Big(\frac{1}{2}\Big) + \frac{5}{18} f\Big(\frac{1}{2} + \frac{\sqrt{3}}{2\sqrt{5}}\Big) + \frac{5}{18} f\Big(\frac{3}{2} - \frac{\sqrt{3}}{2\sqrt{5}}\Big) + \frac{8}{18} f\Big(\frac{3}{2}\Big) + \frac{5}{18} f\Big(\frac{3}{2} + \frac{\sqrt{3}}{2\sqrt{5}}\Big)$$

2 Show that the quadrature formula constructed using the $n + 1$ nodes $x_0, x_1, \ldots, x_n$ can not possibly have degree of precision $2n+2$, i.e. integrate exactly all polynomials of degree up to and including $2n+2$. This implies that the degree of precision, $2n+1$, obtained by Gaussian quadrature is optimal.

Hint: Consider the polynomial M^2, with

$$M(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$$

Possible solution:

Call the interval this quadrature formula applies to for $[a, b]$, where we exclude the trivial case $a = b$ and assume without loss of generality that $b > a$. Now, following the hint it is readily observed that

$$M^2(x) = (x - x_0)^2 (x - x_1)^2 \cdots (x - x_n)^2,$$

is 0 at all the nodes $x_0, x_1, \ldots, x_n$. This implies that for a quadrature formula

$$I(f) = \sum_{k=0}^{n} A_k f(x_k),$$

$I(M^2) = 0$. However M^2 is obviously positive at all points except the nodes and continuous, and so the actual integral of M^2 must be positive, i.e.

$$\int_a^b M^2(x)\,dx > 0.$$

Thus M^2 will not be integrated exactly by the quadrature formula. The proof now follows from the realization that M^2 is a polynomial of degree $2n + 2$.

3 **Cf. Cheney and Kincaid, Exercise 6.1.9**

Assume that you are interpolating the function $f(x) = \sin(100x)$ on the interval $[0, \pi]$ with a linear spline on a uniformly spaced grid. Estimate how many grid points will be required in order to guarantee that the interpolation error is smaller than 10^{-8}?

Possible solution:

Denote by S the interpolating linear spline with grid size $h = \pi/n$. Then we have the estimate

$$\max_{x\in[0,\pi]} |S(x) - f(x)| \leq \frac{h^2}{8} \max_{x\in[0,\pi]} |f''(x)|.$$

Now,

$$f''(x) = -10^4 \sin(100x),$$

and therefore

$$\max_{x\in[0,\pi]} |f''(x)| = 10^4.$$

Thus

$$\max_{x\in[0,\pi]} |S(x) - f(x)| \leq \frac{10^4}{8} h^2.$$

As a consequence, the error is guaranteed to be smaller than 10^{-8}, if

$$h^2 < 8 \times 10^{-12},$$

which, since $h = \pi/n$, is equivalent to the estimate

$$n > \frac{10^6 \pi}{\sqrt{8}}.$$

4 Assume that S is a spline of degree k on $[a, b]$ with $k \geq 2$. Show that S' is a spline of degree $k - 1$ on $[a, b]$.

Possible solution:

Since S is a spline of degree k, it is $k-1$-times continuously differentiable, and therefore S' is $k-2$-times continuously differentiable (continuous in case $k=2$). Furthermore, there exists a partition $a = x_0 < x_1 < \ldots < x_n = b$ such that the restriction of S to each interval (x_{j-1}, x_j), $j = 1, \ldots, n$, is a polynomial of degree k. Hence the restriction of S' to (x_{j-1}, x_j), $j = 1, \ldots, n$, is a polynomial of degree $k-1$, and therefore S' is a spline of degree $k-1$.

$\boxed{5}$ Find the linear function $f(x) = ax + b$ that best approximates the following points:

x	0	1	3	3	2	7
y	-1	1	4	3	3	6

Possible solution:

The optimal linear $f(x) = ax + b$ solves the normal equations

$$a \sum_k x_k^2 + b \sum_k x_k = \sum_k x_k y_k,$$
$$a \sum_k x_k + nb = \sum_k y_k,$$

where n is the number of data points (x_k, y_k). In this case we obtain the equations

$$72a + 16b = 70,$$
$$16a + 6b = 16,$$

with the solution

$$a = \frac{41}{44} \qquad \text{and} \qquad b = \frac{2}{11},$$

that is,

$$f(x) = \frac{41}{44}x + \frac{2}{11}.$$