



- 1 Consider the general iterative method

$$\mathbf{Q}\mathbf{x}^{(k+1)} = (\mathbf{Q} - \mathbf{A})\mathbf{x}^{(k)} + \mathbf{b}$$

with invertible matrices \mathbf{A} , $\mathbf{Q} \in \mathbb{R}^{n \times n}$ for the solution of the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$. Assume that the matrix $\mathbf{I} - \mathbf{Q}^{-1}\mathbf{A}$ has an eigenvalue $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$. Show that there exists a starting vector $\mathbf{x}^{(0)}$ for which the iteration does not converge to a solution of the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Hint: Consider a starting vector of the form $\hat{\mathbf{x}} + \mathbf{u}$, where $\hat{\mathbf{x}}$ is the solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ and \mathbf{u} is an eigenvector of $\mathbf{I} - \mathbf{Q}^{-1}\mathbf{A}$.

Possible solution:

Let $\mathbf{u} \in \mathbb{R}^n \setminus \{0\}$ be an eigenvector of $\mathbf{I} - \mathbf{Q}^{-1}\mathbf{A}$ with respect to some eigenvalue λ with $|\lambda| \geq 1$. Define moreover

$$\mathbf{x}^{(0)} = \mathbf{u} + \mathbf{A}^{-1}\mathbf{b}.$$

We show in the following that

$$\mathbf{x}^{(k)} = \lambda^k \mathbf{u} + \mathbf{A}^{-1}\mathbf{b}. \quad (1)$$

For $k = 0$ this is trivial. Assume now that (1) holds for some $k \geq 0$. Then

$$\begin{aligned} \mathbf{x}^{(k+1)} &= (\mathbf{I} - \mathbf{Q}^{-1}\mathbf{A})\mathbf{x}^{(k)} + \mathbf{Q}^{-1}\mathbf{b} \\ &= (\mathbf{I} - \mathbf{Q}^{-1}\mathbf{A})(\lambda^k \mathbf{u} + \mathbf{A}^{-1}\mathbf{b}) + \mathbf{Q}^{-1}\mathbf{b} \\ &= \lambda^k (\mathbf{I} - \mathbf{Q}^{-1}\mathbf{A})\mathbf{u} + \mathbf{A}^{-1}\mathbf{b} - \mathbf{Q}^{-1}\mathbf{A}\mathbf{A}^{-1}\mathbf{b} + \mathbf{Q}^{-1}\mathbf{b} \\ &= \lambda^{k+1} \mathbf{u} + \mathbf{A}^{-1}\mathbf{b}, \end{aligned}$$

which shows that (1) holds for all k . In particular,

$$\mathbf{x}^{(k)} - \mathbf{A}^{-1}\mathbf{b} = \lambda^k \mathbf{u},$$

which does not tend to 0.

- 2 a) Assume that the matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is symmetric and positive definite. Show that there exists some parameter $\lambda > 0$ such that the damped Richardson iteration

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \frac{1}{\lambda}(\mathbf{A}\mathbf{x}^{(k)} - \mathbf{b})$$

converges to a solution of the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$.

b) Consider now the linear system

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$$

Find a suitable damping parameter λ and compute an approximation of the solution of this system using three steps of damped Richardson iteration.

Possible solution:

a) Damped Richardson iteration converges if (and only if) all eigenvalues of the defining matrix $\mathbf{I} - \frac{1}{\lambda}\mathbf{A}$ are in absolute value smaller than 1. Now denote by

$$\mu_1 \geq \mu_2 \geq \dots \geq \mu_n > 0$$

the eigenvalues of \mathbf{A} (these are all real positive, because \mathbf{A} is assumed to be symmetric and positive definite). Then the eigenvalues of $\mathbf{I} - \frac{1}{\lambda}\mathbf{A}$ are

$$1 - \frac{\mu_1}{\lambda} \leq 1 - \frac{\mu_2}{\lambda} \leq \dots \leq 1 - \frac{\mu_n}{\lambda} < 1.$$

Thus the eigenvalues of $\mathbf{I} - \frac{1}{\lambda}\mathbf{A}$ are *all* in absolute value smaller than 1, if and only if

$$1 - \frac{\mu_1}{\lambda} > -1,$$

which means that

$$\lambda > \frac{\mu_1}{2}. \quad (2)$$

Every parameter $\lambda > 0$ satisfying (2) yields a convergent method.

b) According to part a) of this exercise, we have to choose the damping parameter larger than $\mu_1/2$, where μ_1 is the largest eigenvalue of the matrix of the system. Using the power method we find that $\mu_1 \approx 5.4142$. Thus we can for instance use $\lambda = 3$, which yields the iteration

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \frac{1}{3} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix} \mathbf{x}^{(k)} + \frac{1}{3} \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}.$$

Starting with $\mathbf{x}^{(0)} = [0, 0, 0]^T$, we obtain the iterates

$$\mathbf{x}^{(1)} = \frac{1}{3} \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}, \quad \mathbf{x}^{(2)} = \frac{1}{9} \begin{bmatrix} 7 \\ -17 \\ 8 \end{bmatrix}, \quad \mathbf{x}^{(3)} = \frac{1}{27} \begin{bmatrix} 27 \\ -51 \\ 29 \end{bmatrix}.$$

(The solution is $[1, -2, 1]$.)

3 Compute approximate solutions of the linear system

$$\begin{bmatrix} 3 & -1 & 1 \\ 0 & 5 & 2 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 6 \end{bmatrix}$$

using

- a) three steps of the Jacobi method,
- b) three steps of the Gauss–Seidel method,
- c) three steps of the SOR-method with parameter $\omega = 1.5$.

Use the starting value $\mathbf{x}^{(0)} = [0, 0, 0]^T$ in all cases.

Possible solution:

a) In the Jacobi method, we have the iteration

$$\begin{aligned} \mathbf{x}^{(k+1)} &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}^{(k)} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ -1 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1/3 & -1/3 \\ 0 & 0 & -2/5 \\ 1/3 & 1/3 & 0 \end{bmatrix} \mathbf{x}^{(k)} + \begin{bmatrix} 2 \\ -1/5 \\ 2 \end{bmatrix}. \end{aligned}$$

This yields the iterates

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ -1/5 \\ 2 \end{bmatrix}, \quad \mathbf{x}^{(2)} \approx \begin{bmatrix} 1.2667 \\ -1 \\ 2.6 \end{bmatrix}, \quad \mathbf{x}^{(3)} \approx \begin{bmatrix} 0.8 \\ -1.24 \\ 2.0889 \end{bmatrix}.$$

b) The Gauss–Seidel method reads as

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{3}(x_2^{(k)} - x_3^{(k)} + 6), \\ x_2^{(k+1)} &= \frac{1}{5}(-2x_3^{(k)} - 1), \\ x_3^{(k+1)} &= \frac{1}{3}(x_1^{(k+1)} + x_2^{(k+1)} + 6). \end{aligned}$$

This yields the iterates

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ -0.2 \\ 2.6 \end{bmatrix}, \quad \mathbf{x}^{(2)} \approx \begin{bmatrix} 1.0667 \\ -1.24 \\ 1.9422 \end{bmatrix}, \quad \mathbf{x}^{(3)} \approx \begin{bmatrix} 0.9393 \\ -0.9769 \\ 1.9875 \end{bmatrix}.$$

c) For the SOR method with $\omega = 3/2$ we obtain the iteration

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{2}(x_2^{(k)} - x_3^{(k)} + 6) - \frac{1}{2}x_1^{(k)}, \\ x_2^{(k+1)} &= \frac{3}{10}(-2x_3^{(k)} - 1) - \frac{1}{2}x_2^{(k)}, \\ x_3^{(k+1)} &= \frac{1}{2}(x_1^{(k+1)} + x_2^{(k+1)} + 6) - \frac{1}{2}x_3^{(k)}. \end{aligned}$$

This yields the iterates

$$\mathbf{x}^{(1)} = \begin{bmatrix} 3 \\ -0.3 \\ 4.35 \end{bmatrix}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} -0.825 \\ -2.76 \\ -0.9675 \end{bmatrix}, \quad \mathbf{x}^{(3)} \approx \begin{bmatrix} 2.5162 \\ 1.6605 \\ 5.5721 \end{bmatrix}.$$

Apparently, the SOR method does not converge.

- 4 Implement in MATLAB the Jacobi method for the iterative solution of linear systems.

Test the method on the linear systems of exercises 2 and 3. In addition, test it on the matrix $\mathbf{A} \in \mathbb{R}^{200 \times 200}$ with main diagonal $\mathbf{d} = (4, 4, \dots, 4)$ and lower and upper diagonals $\mathbf{a} = \mathbf{c} = (-1, -1, \dots, -1)$, and the right hand sides $\mathbf{b}_1 = (1, \dots, 1)^T$ and $\mathbf{b}_2 = (1, 2, 3, \dots, 200)^T$.

Note: The Jacobi method in MATLAB will be rather efficient if you implement it using matrix–vector products. The command `diag` can be used both for extracting diagonals of a matrix and constructing diagonal matrices; for instance, `diag(diag(\mathbf{A}))` will produce a diagonal matrix with the same diagonal entries as \mathbf{A} .

Possible solution:

You can find a possible implementation of the method on the webpage of the course.

- 5 Assume that the matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is diagonally dominant. Show that the Jacobi iteration for the solution of the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ converges.

Hint: You may want to show that the matrix $\mathbf{I} - \mathbf{Q}^{-1}\mathbf{A}$ defining the iteration satisfies $\|\mathbf{I} - \mathbf{Q}^{-1}\mathbf{A}\|_\infty < 1$.

Possible solution:

For the Jacobi iteration, the matrix $\mathbf{B} := \mathbf{I} - \mathbf{Q}^{-1}\mathbf{A}$ defining the iteration has the entries

$$b_{ij} = \begin{cases} \frac{a_{ij}}{a_{ii}} & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

Now recall that the ∞ -norm of a matrix $\mathbf{B} \in \mathbb{R}^{n \times n}$ is defined as

$$\|\mathbf{B}\|_\infty := \max_{1 \leq i \leq n} \sum_{j=1}^n |b_{ij}|.$$

For the Jacobi iteration we have

$$\|\mathbf{B}\|_\infty = \max_{1 \leq i \leq n} \sum_{j \neq i} \frac{|a_{ij}|}{|a_{ii}|}.$$

Since the matrix \mathbf{A} is diagonally dominant, we have

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

for all i . Thus

$$\sum_{j \neq i} \frac{|a_{ij}|}{|a_{ii}|} < 1,$$

and therefore also $\|\mathbf{B}\|_\infty < 1$. Thus the Jacobi iteration converges.

6 Use appropriate interpolating polynomials in Lagrange form of degrees one, two and three to approximate each of the following:

- a) $f(8.4)$ if $f(x) = x \ln x$ and the nodes are 8.1, 8.3, 8.6 and 8.7.
 b) $f(-1/3)$ if $f(x) = x^4 - x^3 + x^2 - x + 1$ and the nodes are -0.75 , -0.5 , -0.25 and 0.

Possible solution:

- a) To be accurate, we choose the nodes closest to the x -value where we wish to approximate f . Consequently for the linear case we choose $x_0 = 8.3$ and $x_1 = 8.6$. The polynomial is

$$P_1(x) = f(8.3) \frac{x - 8.6}{-0.3} + f(8.6) \frac{x - 8.3}{0.3}.$$

From this we find

$$P_1(8.4) = 8.3 \ln(8.3) \frac{-0.2}{-0.3} + 8.6 \ln(8.6) \frac{0.1}{0.3} \approx 17.878332.$$

In the quadratic case we add $x_2 = 8.7$ to keep the interval small. The interpolating polynomial is

$$P_2(x) = f(8.3) \frac{(x - 8.6)(x - 8.7)}{0.12} + f(8.6) \frac{(x - 8.3)(x - 8.7)}{-0.03} + f(8.7) \frac{(x - 8.3)(x - 8.6)}{0.04},$$

and we get the approximation

$$P_2(8.4) = 8.3 \ln(8.3) \frac{0.06}{0.12} + 8.6 \ln(8.6) \frac{-0.03}{-0.03} + 8.7 \ln(8.7) \frac{-0.02}{0.04} \approx 17.877160$$

In the cubic case we must use all the points so $x_3 = 8.1$. The interpolating polynomial is

$$P_3(x) = f(8.3) \frac{(x - 8.6)(x - 8.7)(x - 8.1)}{0.024} + f(8.6) \frac{(x - 8.3)(x - 8.7)(x - 8.1)}{-0.015} + f(8.7) \frac{(x - 8.3)(x - 8.6)(x - 8.1)}{0.024} + f(8.1) \frac{(x - 8.3)(x - 8.6)(x - 8.7)}{-0.06},$$

and finally

$$P_3(8.4) = 8.3 \ln(8.3) \frac{0.018}{0.024} + 8.6 \ln(8.6) \frac{-0.009}{-0.015} + 8.7 \ln(8.7) \frac{-0.006}{0.024} + 8.1 \ln(8.1) \frac{0.006}{-0.06} \approx 17.877146.$$

The true value to the digits given is $f(8.4) = 17.877146$.

- b) The procedure is analogue to the previous task. The results are summarized in the table below.

n	x_0, x_1, \dots, x_n	$P_n(-1/3)$
1	-0.5, -0.25	1.5338542
2	-0.5, -0.25, 0.0	1.5034722
3	-0.5, -0.25, 0.0, -0.75	1.4918981

The true value is $f(-1/3) = 1.4938272$ to the digits given.

- 7 Use Neville's algorithm to obtain the approximations for the previous exercise

Possible solution:

- a) We number the nodes in ascending order $x_0 = 8.1, x_1 = 8.3, x_2 = 8.6$ and $x_3 = 8.7$ for $f(x) = x \ln x$, and compute the table of approximations $S_{ij}(x)$ using the Neville 2nd recurrence relation:

$$S_{ij}(x) = \frac{x - x_{i-j}}{x_i - x_{i-j}} S_{i,j-1}(x) + \frac{x_i - x}{x_i - x_{i-j}} S_{i-1,j-1}(x)$$

at $x = 8.4$.

8.1	16.944099			
8.3	17.564921	17.875332		
8.6	18.505155	17.878332	17.877132	
8.7	18.820910	17.873644	17.877160	17.877146

Here $P_1(8.4), P_2(8.4), P_3(8.4)$ from the previous task corresponds to $S_{21}(8.4), S_{32}(8.4), S_{33}(8.4)$ respectively, and the computed values are seen to be the same.

- b) We again number the nodes in ascending order $x_0 = -0.75, x_1 = -0.5, x_2 = -0.25$ and $x_3 = 0$ with $f(x) = x^4 - x^3 + x^2 - x + 1$, and compute the table of approximations $S_{ij}(x)$ at $x = -1/3$

-0.75	3.0507812			
-0.5	1.9375	1.1953125		
-0.25	1.3320312	1.5338542	1.4774306	
0	1	1.4427083	1.5034722	1.4918981

Again $P_1(-1/3), P_2(-1/3), P_3(-1/3)$ from the previous task corresponds to $S_{21}(-1/3), S_{32}(-1/3), S_{33}(-1/3)$ respectively, and the computed values are seen to match.