Norwegian University of Science
and Technology
Department of Mathematics

MA2501 Numerical Methods
Spring 2016

Exercise set 11

1 Consider the initial value problem

$$\begin{aligned} x' &= \sin(t^2 + x), \\ x(0) &= 0. \end{aligned}$$

a) Use Euler's method with a step size of $h = 1/4$ in order to obtain an approximation of $x(2)$.

b) Use the improved Euler method (Heun's second order method) with a step size of $h = 1/2$ in order to obtain an approximation of $x(2)$.

c) Use the classical Runge–Kutta method with a step size of $h = 1$ in order to obtain an approximation of $x(2)$.

2 Assume that the function $f \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and decreasing in its second component. That is,

$$f(t, x) \le f(t, z) \qquad \text{whenever} \qquad x \ge z.$$

Show that the implicit Euler method for the solution of the differential equation

$$\begin{aligned} x' &= f(t, x), \\ x(t_0) &= x_0, \end{aligned}$$

is well-defined. That is, regardless of the step-size $h > 0$, the (non-linear) equation that has to be solved in each iteration has a unique solution.

3 Compute three steps with step size $h = 1$ for the numerical solution of the differential equation

$$\begin{aligned} x' &= -3x - e^x, \\ x(0) &= 1, \end{aligned}$$

using:

a) Euler's method.

b) The improved Euler method.

c) The implicit Euler method. Numerically solve the non-linear equations you obtain by performing two steps of Newton's method in each step of the implicit Euler method.

4 The third order Adams–Bashforth method has the form

$$\begin{aligned}
x_{k+1} &= x_k + h\Big(\frac{23}{12}f_k - \frac{16}{12}f_{k-1} + \frac{5}{12}f_{k-2}\Big),\\
t_{k+1} &= t_k + h,\\
f_{k+1} &= f(t_{k+1}, x_{k+1}).
\end{aligned}$$

Show that this method can be derived by interpolating the function $\tau \mapsto f(\tau, x(\tau))$ in the points t_{k-2}, t_{k-1}, and t_k and then integrating the resulting quadratic polynomial.

5 Consider the differential equation

$$\begin{aligned}
x' &= x - \frac{x^2}{2},\\
x(0) &= 1.
\end{aligned}$$

Compute three steps with step size $h = 1$ using:

a) The second order Adams–Bashforth method, which for an autonomous ODE is defined by,

$$x_{k+1} = x_k + h\Big(\frac{3}{2}f_k - \frac{1}{2}f_{k-1}\Big),$$

where $f_k \equiv f(x_k)$.

b) The second order Adams–Moulton method, which for an autonomous ODE is (implicitly) defined by

$$x_{k+1} = x_k + h\Big(\frac{1}{2}f_{k+1} + \frac{1}{2}f_k\Big).$$

c) The second order Adams–Bashforth–Moulton method.

In all three cases, use the improved Euler method for computing the iterate x_1.

6 Consider the second order initial value problem

$$\begin{aligned}
x'' &= -\sin(x) + x',\\
x(0) &= 0,\\
x'(0) &= 1.
\end{aligned}$$

a) Rewrite the second order equation as a system of first order equations.

b) Compute two steps of the classical Runge–Kutte method with step size $h = 1/2$ in order to obtain an approximation of $x(1)$.

7 **a)** Construct the free cubic spline for the following data.

x	$f(x)$
0.1	−0.62049958
0.2	−0.28398668
0.3	0.00660095
0.4	0.24842440

b) The data in the previous task were constructed using the function

$$f(x) = x\cos x - 2x^2 + 3x - 1$$

Use the cubic spline to approximate $f(0.25))$ and $f'(0.25)$ and calculate the actual error in both cases.