MA2501 Numerical Methods
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Solutions to exercise set 7

1 Cf. Cheney and Kincaid, Exercise 4.1.9

Consider the data points

x_i	0	1	2	4	6
$f(x_i)$	1	9	23	93	259

a) Find the interpolation polynomial through these data points using Newton interpolation, and compute an approximation of f at $x = 3$.

b) Do the same using only the first four interpolation points.

Possible solution:

a) We obtain the following table for the divided differences:

0	1				
		8			
1	9		3		
		14		1	
2	23		7		0
		35		1	
4	93		12		
		83			
6	259				

The interpolation polynomial (in nested Newton form) is therefore

$$p(x) = 1 + x(8 + (x-1)(3 + (x-2) + 0 \cdot (x-4))).$$

(Obviously, one can omit the term $0 \cdot (x-4)$ in that expression). Evaluating this polynomial, we see that $p(3) = 49$.

b) For the interpolation polynomial of degree three through the first four interpolation points, we can simply reuse the previously computed table of divided differences and read off the result from there using only the first four columns. In this particular case, we obtain the same interpolation polynomial p as before (therefore, obviously, also its value at the point $x = 3$ is the same as above).

2 Suppose we have the nodes

$$x_0 = -2,\ x_1 = -1,\ x_2 = 1 \text{ and } x_3 = 4,$$

and know the divided differences

$$f[x_3] = 11,\ f[x_2, x_3] = 5,\ f[x_2, x_0, x_1] = -2, \text{ and } f[x_0, x_2, x_1, x_3] = 0.6.$$

What is f(-1)?

Possible solution:

To start we exploit the fact that divided differences are invariant under permutations of the nodes. Thus, sorting the nodes in increasing order, we have:

$$f[x_3] = 11,\ f[x_2, x_3] = 5,\ f[x_0, x_1, x_2] = -2, \text{ and } f[x_0, x_1, x_2, x_3] = 0.6.$$

We seek $f(-1) = f(x_1) = f[x_1]$. From the recursive definition of divided differences:

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

which rearranged to isolate the unknown $f[x_1, x_2, x_3]$ becomes

$$f[x_1, x_2, x_3] = f[x_0, x_1, x_2] + (x_3 - x_0) f[x_0, x_1, x_2, x_3] = -2 + (4 - (-2))0.6 = 1.6.$$

Similarly we find $f[x_1, x_2]$ as

$$f[x_1, x_2] = f[x_2, x_3] - (x_3 - x_1) f[x_1, x_2, x_3] = 5 - (4 - (-1))1.6 = -3,$$

$f[x_2]$ as

$$f[x_2] = f[x_3] - (x_3 - x_2) f[x_2, x_3] = 11 - (4 - 1)5 = -4,$$

and finally $f[x_1] = f(-1)$ as

$$f[x_1] = f[x_2] - (x_2 - x_1) f[x_1, x_2] = -4 - (1 - (-1))(-3) = 2.$$

3 Use the interpolation error formula to find a bound for the error, and compare the bound to the actual error for the case $n = 2$ for Exercise 6 in Exercise set 6.

Possible solution:

a) For $f(x) = x \ln x$ we first calculate $f'(x) = \ln x + 1$, $f''(x) = 1/x$ and $f^{(3)}(x) = -1/x^2$. Since $|f^{(3)}(x)| = 1/x^2$ is a strictly decreasing function of x for positive x we find a bound M on the interval $(8.3, 8.7)$ for $n = 2$ by $M = 1/8.3^2$. Employing the First Interpolation Error Theorem in the textbook with $n = 2$ and $x = 8.4$ on this interval gives the error bound:

$$\begin{aligned} |f(8.4) - p_2(8.4)| &\leq \frac{1}{(2+1)!}\frac{1}{8.3^2}|(8.4-8.3)||(8.4-8.6)||(8.4-8.7)| \\ &= \frac{0.1 \cdot 0.2 \cdot 0.3}{6 \cdot 8.3^2} \leq 1.46 \times 10^{-5}. \end{aligned}$$

The true error is 1.38×10^{-5} correct to the digits given. As expected this is lower than the error bound.

b) For $f(x) = x^4 - x^3 + x^2 - x + 1$ we again calculate $f'(x) = 4x^3 - 3x^2 + 2x - 1$, $f''(x) = 12x^2 - 6x + 2$ and $f^{(3)}(x) = 24x - 6$. Since $|f^{(3)}(x)| = |24x - 6|$ is easily observed to be a strictly decreasing function of x for negative x, we find a bound M on the interval $(-0.5, 0)$ for $n = 2$ by $M = |f^{(3)}(-0.5)| = 18$. Employing the First Interpolation Error Theorem in the textbook with $n = 2$ and $x = -1/3$ on this interval gives the error bound:

$$\begin{aligned} |f(-1/3) - p_2(-1/3)| &\leq \frac{1}{(2+1)!}18|(-1/3+1/2)||(-1/3+1/4)||(-1/3-0)| \\ &= \frac{18}{6 \cdot 6 \cdot 12 \cdot 3} = \frac{1}{72} \leq 1.4 \times 10^{-2}. \end{aligned}$$

The true error is 9.6×10^{-3} correct to the digits given. As expected this is lower than the error bound.

4 Given the function $f(x) = e^x \sin x$ on the interval $[-4, 2]$.

a) Show by induction

$$f^{(m)}(x) = \frac{d^m}{dx^m} f(x) = 2^{m/2} e^x \sin(x + m\pi/4).$$

b) Let $p_n(x)$ be the polynomial that interpolates $f(x)$ in $n+1$ uniformly distributed nodes (including the endpoints). Find an upper bound for the interpolation error on this interval, expressed using n. What must n be to guarantee an error less than 10^{-5}? Use trial and error or calculate it using Matlab.

c) Use Matlab to verify that the value of n found in **b)**, indeed results in an error less than 10^{-5}.

Possible solution:

a) It is trivial to see that the formula is valid for $m = 0$. Assume now it is correct for some $m \geq 0$. Then

$$\begin{aligned}
f^{(m+1)}(x) = \frac{d}{dx} f^{(m)}(x) &= 2^{m/2} \left[e^x \sin(x + m\pi/4) + e^x \cos(x + m\pi/4) \right] \\
&= 2^{m/2} e^x \left[\sin(x + m\pi/4) + \cos(x + m\pi/4) \right] \\
&= 2^{m/2} e^x \sqrt{2} \left[\frac{1}{\sqrt{2}} \sin(x + m\pi/4) + \frac{1}{\sqrt{2}} \cos(x + m\pi/4) \right] \\
&= 2^{m/2} e^x 2^{1/2} \left[\cos(\pi/4) \sin(x + m\pi/4) + \sin(\pi/4) \cos(x + m\pi/4) \right] \\
&= 2^{(m+1)/2} e^x \sin((x + m\pi/4) + \pi/4) \\
&= 2^{(m+1)/2} e^x \sin(x + (m+1)\pi/4).
\end{aligned}$$

This shows that the formula holds for $m + 1$ as well, assuming it holds for m. The proof by induction is then complete, and the stated formula is seen to hold for all $m \geq 0$

b) Our function f has continous derivatives of any order on the entire real line and on the interval $[-4, 2]$ it satisfies

$$|f^{(n+1)}(x)| = 2^{(n+1)/2} e^x |\sin(x + (n+1)\pi/4)| \leq 2^{(n+1)/2} e^2 = M,$$

where we have used that e^x is a strictly increasing positiv function and that $|\sin(x)| \leq 1$ for all x. The Second Interpolation Error Theorem is applicable and we have for $x \in [-4, 2]$

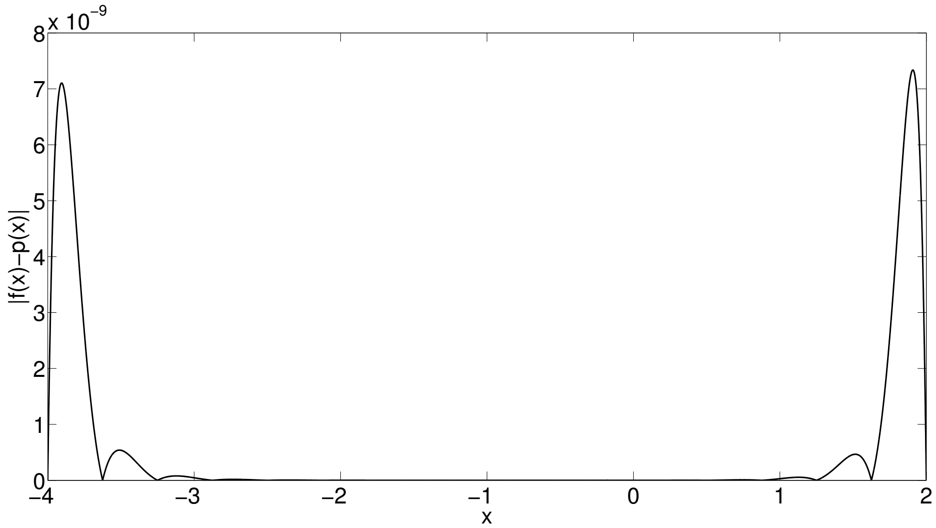
$$\begin{aligned}
|f(x) - p(x)| \leq \frac{1}{4(n+1)} M h^{n+1} &= \frac{1}{4(n+1)} 2^{(n+1)/2} e^2 \left(\frac{6}{n} \right)^{n+1} \\
&= \frac{e^2}{4(n+1)} \left(\frac{6\sqrt{2}}{n} \right)^{n+1}.
\end{aligned}$$

By differentiating, this expression is readliy seen to decrease with increasing positive n. Using trial and error we find that the expression has the value 1.27×10^{-5} for $n = 15$ and 2.26×10^{-6} for $n = 16$. Thus $n = 16$ is required to guarantee an accuracy of 10^{-5}.

c) Using MATLAB we have plotted below $|f(x) - p(x)|$ over the interval, where $p(x)$ is the interpolating polynomial from *b)* with $n = 16$.

We observe that the error stays several orders of magnitude lower than our requirement of 10^{-5}. This is to be expected. Recall that we were in fact guaranteed an error less than 2.26×10^{-6}, and the theorem only gives fairly loose upper bounds.

Also observe that the errors near the endpoints dominate the errors towards the middle of the interval. This is typical, and illustrates why Chebyshev nodes, which become more clustered as we approach the endpoints, usually gives siginificantly smaller errors for the same number of nodes.



5 Consider the function $f(x) = (x^2 + 1)e^x$.

a) Use central differences, i.e., the formula

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h},$$

with step sizes $h = 1$, $h = 1/2$, $h = 1/4$, $h = 1/8$, in order to approximate $f'(0)$.

b) Use Richardson extrapolation for obtaining a better approximation of $f'(0)$ from the values you have already computed.

Possible solution:

a) The approximations of $f'(0)$ for the different step sizes are:

$$\begin{aligned} h = 1: &\quad 2.350402387287603, \\ h = \frac{1}{2}: &\quad 1.302738263734369, \\ h = \frac{1}{4}: &\quad 1.073602346434715, \\ h = \frac{1}{8}: &\quad 1.018271923834063. \end{aligned}$$

b)
- The first step in Richardson extrapolation uses the approximations of $f'(0)$ calculated above. These were:

$$\begin{aligned} D(0,0) &= 2.350402387287603, \\ D(1,0) &= 1.302738263734369, \\ D(2,0) &= 1.073602346434715, \\ D(3,0) &= 1.018271923834063. \end{aligned}$$

- In the next step, we compute the next "column" by the formula

$$D(k,1) = D(k,0) + \frac{1}{3}\big(D(k,0) - D(k-1,0)\big)$$

for $k \geq 1$. We obtain

$$D(1,1) = 0.953516889216624,$$
$$D(2,1) = 0.997223707334830,$$
$$D(3,1) = 0.999828449633845.$$

- Next we compute

$$D(k,2) = D(k,1) + \frac{1}{15}\big(D(k,1) - D(k-1,1)\big)$$

for $k \geq 2$, obtaining

$$D(2,2) = 1.000137495209377,$$
$$D(3,2) = 1.000002099120446.$$

- Finally, we have

$$D(3,3) = D(3,2) + \frac{1}{63}\big(D(3,2) - D(2,2)\big)$$

and thus

$$D(3,3) = 0.999999949976177.$$

(Note that the actual solution is $f'(0) = 1$. Thus the approximations indeed become significantly better.)

6 Derive an $\mathcal{O}(h^4)$ five point formula to approximate $f'(x_0)$ that uses $f(x_0-h)$, $f(x_0)$, $f(x_0+h)$, $f(x_0+2h)$ and $f(x_0+3h)$. Test the formula on $f(x) = \sin x$ at $x = 1$ and try to verify numerically that is has the stated order.

Hint: Consider the expression $Af(x_0-h)+Bf(x_0+h)+Cf(x_0+2h)+Df(x_0+3h)$. Expand in fourth Taylor polynomials, and choose A, B, C and D appropriately.

Possible solution:

We follow the hint and expand the terms in $Af(x_0-h)+Bf(x_0+h)+Cf(x_0+2h)+Df(x_0+3h)$ in fourth order Taylor polynomials and reorganize.

$$
\begin{aligned}
&Af(x_0-h)+Bf(x_0+h)+Cf(x_0+2h)+Df(x_0+3h) = \\
&= A\left[f(x_0)-hf'(x_0)+\frac{h^2}{2!}f''(x_0)-\frac{h^3}{3!}f^{(3)}(x_0)+\frac{h^4}{4!}f^{(4)}(x_0)-\frac{h^5}{5!}f^{(5)}(\xi_A)\right] \\
&+ B\left[f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f^{(3)}(x_0)+\frac{h^4}{4!}f^{(4)}(x_0)+\frac{h^5}{5!}f^{(5)}(\xi_B)\right] \\
&+ C\left[f(x_0)+2hf'(x_0)+\frac{(2h)^2}{2!}f''(x_0)+\frac{(2h)^3}{3!}f^{(3)}(x_0)+\frac{(2h)^4}{4!}f^{(4)}(x_0)+\frac{(2h)^5}{5!}f^{(5)}(\xi_C)\right] \\
&+ D\left[f(x_0)+3hf'(x_0)+\frac{(3h)^2}{2!}f''(x_0)+\frac{(3h)^3}{3!}f^{(3)}(x_0)+\frac{(3h)^4}{4!}f^{(4)}(x_0)+\frac{(3h)^5}{5!}f^{(5)}(\xi_D)\right] \\
&= (A+B+C+D)f(x_0)+(-A+B+2C+3D)hf'(x_0) \\
&+ (A+B+4C+9D)\frac{h^2}{2!}f''(x_0)+(-A+B+8C+27D)\frac{h^3}{3!}f^{(3)}(x_0) \\
&+ (A+B+16C+81D)\frac{h^4}{4!}f^{(4)}(x_0)+(-A+B+32C+243D)4\frac{h^5}{5!}f^{(5)}(\xi)
\end{aligned}
$$

Where $f^{(5)}(\xi)=[f^{(5)}(\xi_A)+f^{(5)}(\xi_B)+f^{(5)}(\xi_C)+f^{(5)}(\xi_D)]/4$ for some $\xi\in(x_0-h,x_0+3h)$. To get a $\mathcal{O}(h^4)$ five point approximation for $f'(x_0)$ we require the coefficient in front of $f'(x_0)$ to equal 1 and the $\mathcal{O}(h^2)$-, $\mathcal{O}(h^3)$- and $\mathcal{O}(h^4)$-term to vanish. This yields the linear system of equations

$$
\begin{aligned}
h(-A+B+2C+3D) &= 1,\\
A+B+4C+9D &= 0,\\
-A+B+8C+27D &= 0,\\
A+B+16C+81D &= 0,
\end{aligned}
$$

which is easily solved for

$$A=-\frac{3}{12h},\quad B=\frac{18}{12h},\quad C=-\frac{6}{12h},\quad D=\frac{1}{12h}.$$

Inserting this into the expression and moving the term

$$(A+B+C+D)f(x_0)=\frac{10}{12h}f(x_0)$$

to the left hand side we get

$$
\begin{aligned}
&\frac{-3f(x_0-h)-10f(x_0)+18f(x_0+h)-6f(x_0+2h)+f(x_0+3h)}{12h}\\
&=f'(x_0)+2.4h^4f^{(5)}(\xi)=f'(x_0)+\mathcal{O}(h^4),
\end{aligned}
$$

so the expression on the left hand side is the desired five point formula.

To verify that this formula has indeed fourth order, we compute with it an approximation to the derivative of $f(x)=\sin x$ at $x=1$ for $h=1/2^i$ for $i=2,3,\dots,9$, and plot the

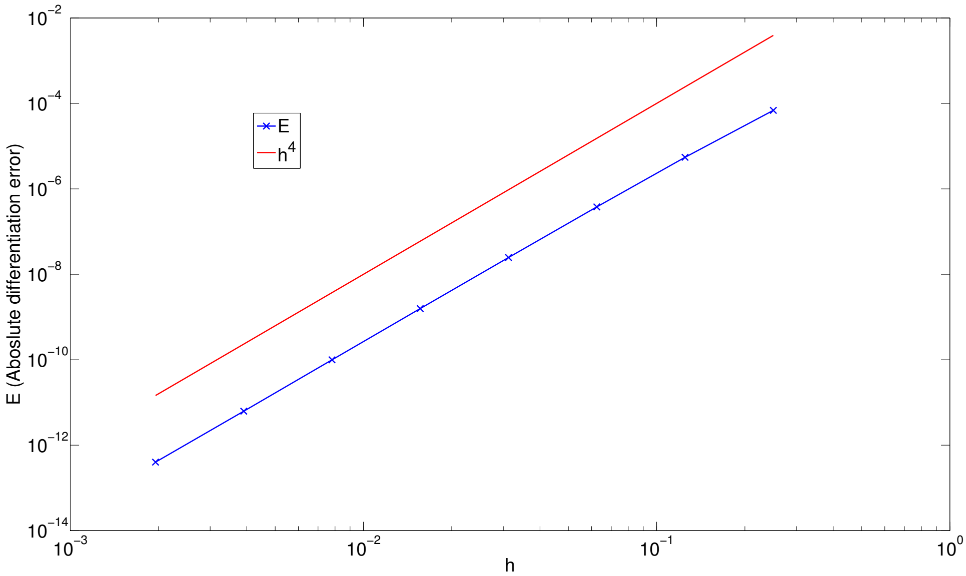


Figure 1:

absolute error, i.e. the abolute value of the difference between the computed value and $f'(1) = \cos 1$, against h with logarithmic axes in MATLAB, see Figure 1. We include a plot of the function h^4 (marked in red) for comparison.

If the formula is indeed fourth order, the absolute error should be $\mathcal{O}(h^4)$. This means that it behaves like the function ch^4 for some positive constant c and sufficiently small values of h. Denote this error by E. If $E \approx ch^4$ for small h then taking logarithms we see that $\log E \approx 4 \log h + \log c$. So if the formula is fourth order, our logarithmic error plot should be a straight line with the same slope as the plot of the function h^4. We observe precisely this. Note that reducing h further would cause the line to flatten out, since numerical rounding errors would then start to dominate.

7 Cf. Cheney and Kincaid, Exercises 4.1.16–17

Assume that the function φ has the form

$$\varphi(h) = L - c_1 h - c_2 h^2 - c_3 h^3 - c_4 h^4 - \dots$$

a) Combine the values $\varphi(h)$ and $\varphi(h/2)$ in order to obtain a higher order approximation of L.

b) Try to generalize the idea of Richardson extrapolation to the function φ.

Possible solution:

a) We have

$$\begin{aligned}\varphi(h) =& L - c_1 h - c_2 h^2 - c_3 h^3 - c_4 h^4 - \dots,\\ \varphi(h/2) =& L - c_1 \frac{h}{2} - c_2 \frac{h^2}{4} - c_3 \frac{h^3}{8} - c_4 \frac{h^4}{16} - \dots\end{aligned}$$

Consequently,

$$2\varphi(h/2) - \varphi(h) = L + c_2 \frac{h^2}{2} + c_3 \frac{3h^3}{4} + c_4 \frac{7h^4}{8} + \dots,$$

which is an approximation of L of order $\mathcal{O}(h^2)$.

b) Assume now that

$$\psi(h) = L - a_m h^m - a_{m+1} h^{m+1} - a_{m+2} h^{m+2} - \dots$$

is an order $\mathcal{O}(h^m)$ approximation of L. Then

$$\psi(h/2) = L - a_m \frac{h^m}{2^m} - a_{m+1} \frac{h^{m+1}}{2^{m+1}} - a_{m+2} \frac{h^{m+2}}{2^{m+2}} - \dots,$$

and therefore

$$2^m \psi(h/2) - \psi(h) = (2^m - 1)L + a_{m+1} \frac{h^{m+1}}{2} + a_{m+1} \frac{3h^{m+2}}{4} + \dots$$

Thus

$$\psi(h/2) + \frac{1}{2^m - 1}\big(\psi(h/2) - \psi(h)\big) = L + d_{m+1} h^{m+1} + d_{m+1} h^{m+2} + \dots$$

is an order $\mathcal{O}(h^{m+1})$ approximation of L. Thus is makes sense to compute approximations of L by:

- Compute approximations

$$D(k, 0) := \phi(2^{-k} h).$$

(This is the same as for the usual Richardson extrapolation.)

- Compute recursively

$$D(k, m) := D(k, m-1) + \frac{1}{2^m - 1}\big(D(k, m-1) - D(k-1, m-1)\big).$$

(This is almost the same as the usual Richardson extrapolation, but the factor $1/(4^m - 1)$ is replaced by $1/(2^m - 1)$.)

8 Write a Matlab-program for the approximation of derivatives using central differences and Richardson extrapolation. Your program should take as an input a function f, a point x where you want to approximate f', a basic step size h, and the desired approximation order.

Test your program on the function sin with $x = \pi/3$ and on the function from exercise 5. For which parameters do you obtain the best results?

Possible solution:

There is an example program on the webpage of the course.

- Applying the program to the function sin with $x = \pi/3$ and $h = 1$, it turns out that the error becomes smallest for $k = 5$ (that is, $D(5, 5)$ is the best approximation of the true value $\sin'(\pi/3) = 1/2$).
- In the case of the function $f(x) = (x^2 + 1)e^x$ with $x = 0$ and $h = 1$, the best result is obtained for $k = 7$. In Figure 2 you find a plot of $|D(k, k) - f'(0)|$ versus k for different initial step sizes h. In all cases one observes first a rapid decrease of the error as k increases, followed by a slow (and unsteady) increase of the error. The initial decrease is due to the higher approximation order. At some point, however, rounding errors due to the limited precision of the computations become notable, and the error starts to increase again.

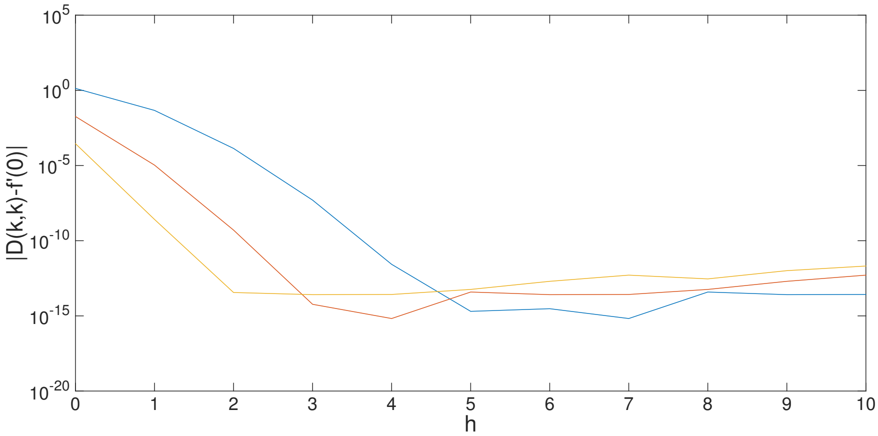


Figure 2: $h = 1$ blue, $h = 1/8$ red, $h = 1/64$ yellow.