Norwegian University of Science and Technology
Department of Mathematics

MA2501 Numerical Methods
Spring 2015

Exercise set 7

1 **Cf. Cheney and Kincaid, Exercise 4.1.9**

Consider the data points

x_i	0	1	2	4	6
$f(x_i)$	1	9	23	93	259

a) Find the interpolation polynomial through these data points using Newton interpolation, and compute an approximation of f at $x = 3$.

b) Do the same using only the first four interpolation points.

2 Suppose we have the nodes

$$x_0 = -2,\ x_1 = -1,\ x_2 = 1 \text{ and } x_3 = 4,$$

and know the divided differences

$$f[x_3] = 11,\ f[x_2, x_3] = 5,\ f[x_2, x_0, x_1] = -2, \text{ and } f[x_0, x_2, x_1, x_3] = 0.6.$$

What is f(-1)?

3 Use the interpolation error formula to find a bound for the error, and compare the bound to the actual error for the case $n = 2$ for task 6 in Exercise 6.

4 Given the function $f(x) = e^x \sin x$ on the interval $[-4, 2]$.

a) Show by induction

$$f^{(m)}(x) = \frac{d^m}{dx^m} f(x) = 2^{m/2} e^x \sin(x + m\pi/4).$$

b) Let $p_n(x)$ be the polynomial that interpolates $f(x)$ in $n+1$ uniformly distributed nodes (including the endpoints). Find an upper bound for the interpolation error on this interval, expressed using n. What must n be to guarantee an error less than 10^{-5}? Use trial and error or calculate it using MATLAB.

c) Use MATLAB to verify that the value of n found in **b)**, indeed results in an error less than 10^{-5}.

5 Consider the function $f(x) = (x^2 + 1)e^x$.

a) Use central differences, i.e., the formula

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h},$$

with step sizes $h = 1$, $h = 1/2$, $h = 1/4$, $h = 1/8$, in order to approximate $f'(0)$.

b) Use Richardson extrapolation for obtaining a better approximation of $f'(0)$ from the values you have already computed.

6 Derive an $\mathcal{O}(h^4)$ five point formula to approximate $f'(x_0)$ that uses $f(x_0 - h)$, $f(x_0)$, $f(x_0 + h)$, $f(x_0 + 2h)$ and $f(x_0 + 3h)$. Test the formula on $f(x) = \sin x$ at $x = 1$ and try to verify numerically that is has the stated order.

Hint: Consider the expression $Af(x_0 - h) + Bf(x_0 + h) + Cf(x_0 + 2h) + Df(x_0 + 3h)$. Expand in fourth Taylor polynomials, and choose A, B, C and D appropriately.

7 **Cf. Cheney and Kincaid, Exercises 4.1.16–17**

Assume that the function φ has the form

$$\varphi(h) = L - c_1 h - c_2 h^2 - c_3 h^3 - c_4 h^4 - \ldots$$

a) Combine the values $\varphi(h)$ and $\varphi(h/2)$ in order to obtain a higher order approximation of L.

b) Try to generalize the idea of Richardson extrapolation to the function φ.

8 Write a MATLAB-program for the approximation of derivatives using central differences and Richardson extrapolation. Your program should take as an input a function f, a point x where you want to approximate f', a basic step size h, and the desired approximation order.

Test your program on the function sin with $x = \pi/3$ and on the function from exercise 5. For which parameters do you obtain the best results?