Norwegian University of Science and Technology
Department of Mathematics

MA2501 Numerical Methods
Spring 2015

Exercise set 6

1 Consider the general iterative method

$$\mathbf{Q}\mathbf{x}^{(k+1)} = (\mathbf{Q} - \mathbf{A})\mathbf{x}^{(k)} + \mathbf{b}$$

with invertible matrices $\mathbf{A}$, $\mathbf{Q} \in \mathbb{R}^{n\times n}$ for the solution of the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$. Assume that the matrix $\mathbf{I} - \mathbf{Q}^{-1}\mathbf{A}$ has an eigenvalue $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$. Show that there exists a starting vector $\mathbf{x}^{(0)}$ for which the iteration does not converge to a solution of the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Hint: Consider a starting vector of the form $\hat{\mathbf{x}} + \mathbf{u}$, where $\hat{\mathbf{x}}$ is the solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{u}$ is an eigenvector of $\mathbf{I} - \mathbf{Q}^{-1}\mathbf{A}$.

2 **a)** Assume that the matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is symmetric and positive definite. Show that there exists some parameter $\lambda > 0$ such that the damped Richardson iteration

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \frac{1}{\lambda}(\mathbf{A}\mathbf{x}^{(k)} - \mathbf{b})$$

converges to a solution of the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$.

b) Consider now the linear system

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$$

(Note that the matrix of this system has already appeared in exercise **??**.) Find a suitable damping parameter λ and compute an approximation of the solution of this system using three steps of damped Richardson iteration.

3 Compute approximate solutions of the linear system

$$\begin{bmatrix} 3 & -1 & 1 \\ 0 & 5 & 2 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 6 \end{bmatrix}$$

using

a) three steps of the Jacobi method,

b) three steps of the Gauss–Seidel method,

c) three steps of the SOR-method with parameter $\omega = 1.5$.

Use the starting value $\mathbf{x}^{(0)} = (0, 0, 0)^T$ in all cases.

4 Implement in MATLAB the Jacobi method for the iterative solution of linear systems.

Test the method on the linear systems of exercises 2 and 3. In addition, test it on the matrix $\mathbf{A} \in \mathbb{R}^{200\times 200}$ with main diagonal $\mathbf{d} = [4, 4, \ldots, 4]$ and lower and upper diagonals $\mathbf{a} = \mathbf{c} = [-1, -1, \ldots, -1]$, and the right hand sides $\mathbf{b}_1 = [1, \ldots, 1]^T$ and $\mathbf{b}_2 = [1, 2, 3, \ldots, 200]^T$.

Note: The Jacobi method in MATLAB will be rather efficient if you implement it using matrix–vector products. The command `diag` can be used both for extracting diagonals of a matrix and constructing diagonal matrices; for instance, `diag(diag(A))` will produce a diagonal matrix with the same diagonal entries as $\mathbf{A}$.

5 Assume that the matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is diagonally dominant. Show that the Jacobi iteration for the solution of the system $\mathbf{Ax} = \mathbf{b}$ converges.

Hint: You may want to show that the matrix $\mathbf{I} - \mathbf{Q}^{-1}\mathbf{A}$ defining the iteration satisfies $\|\mathbf{I} - \mathbf{Q}^{-1}\mathbf{A}\|_\infty < 1$.

6 Use appropriate interpolating polynomials in Lagrange form of degrees one, two and three to approximate each of the following:

a) $f(8.4)$ if $f(x) = x \ln x$ and the nodes are 8.1, 8.3, 8.6 and 8.7.

b) $f(-1/3)$ if $f(x) = x^4 - x^3 + x^2 - x + 1$ and the nodes are -0.75, -0.5, -0.25 and 0.

7 Use Neville's algorithm to obtain the approximations for the previous exercise