FIXED POINT ITERATIONS

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1. Fixed Point Iteration for Non-linear Equations

Our goal is the solution of an equation

$$F(x) = 0, \tag{1}$$

where $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector valued mapping in n variables. Since the target space is the same as the domain of the mapping F, one can equivalently rewrite this as

$$x = x + F(x).$$

More general, if $G\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is any matrix valued function such that $G(x)$ is invertible for every $x \in \mathbb{R}^n$, then (1) is equivalent to the equation

$$G(x)F(x) = 0,$$

which in turn is equivalent to

$$x = x + G(x)F(x).$$

Setting

$$\Phi(x) := x + G(x)F(x),$$

it follows that x solves (1), if and only if x solves $\Phi(x) = x$; that is, x is a fixed point of Φ.

A simple idea for the solution of fixed point equations is that of *fixed point iteration*. Starting with any point $x^{(0)} \in \mathbb{R}^n$ (which, preferably, should be an approximation of a solution of (1)), one defines the sequence $(x^{(k)})_{k\in\mathbb{N}} \subset \mathbb{R}^n$ by

$$x^{(k+1)} := \Phi(x^{(k)}).$$

The rationale behind that definition is the fact that this sequence will become stationary after some index k, if (and only if) $x^{(k)}$ is a fixed point of Φ. Moreover, one can assume that $x^{(k)}$ is close to a fixed point, if $x^{(k)} \approx x^{(k+1)}$. In the following we will investigate conditions that guarantee that this iteration actually works. To that end, we will have to study (shortly) the concept of matrix norms.

2. Matrix Norms

Recall that a *vector norm* on $\mathbb{R}^n$ is a mapping $\|\cdot\|\colon \mathbb{R}^n \to \mathbb{R}$ satisfying the following conditions:

- $\|x\| > 0$ for $x \neq 0$.
- $\|\lambda x\| = |\lambda|\|x\|$ for $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.
- $\|x + y\| \leq \|x\| + \|y\|$ for all x, $y \in \mathbb{R}^n$.

Since the space $\mathbb{R}^{n\times n}$ of all matrices is also a vector space, it is also possible to consider norms there. In contrast to usual vectors, it is, however, also possible to multiply matrices (that is, the matrices form not only a vector space but also an *algebra*). One now calls a norm on $\mathbb{R}^{n\times n}$ a *matrix norm*, if the norm behaves well under multiplication of matrices. More precisely, we require the norm to satisfy (in addition) the condition:

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- $\|AB\| \le \|A\|\|B\|$ for all A, $B \in \mathbb{R}^{n\times n}$.

This condition is called the *sub-multiplicativity* of the norm.

Assume now that we are given a norm $\|\cdot\|$ on $\mathbb{R}^n$. Then this norm *induces* a norm on $\mathbb{R}^{n\times n}$ by means of the formula

$$\|A\| := \sup\bigl\{\|Ax\| : \|x\| \le 1\bigr\}.$$

That is, the norm measures the maximal "elongation" of a vector x when it is multiplied by the matrix A. Such a norm is interchangeably referred to as either the matrix norm *induced by* $\|\cdot\|$ or the matrix norm *subordinate to* $\|\cdot\|$.

One can show that a subordinate matrix norm always has the following properties:

(1) We always have $\|\mathrm{Id}\| = 1$.
(2) A subordinate matrix norm is always sub-multiplicative.
(3) If A is invertible, then $\|A\|\|A^{-1}\| \ge 1$.
(4) For every $A \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$ we have the inequality

$$\|Ax\| \le \|A\|\|x\|.$$

Even more, one can write $\|A\|$ as

$$\|A\| = \inf\bigl\{C > 0 : \|Ax\| \le C\|x\| \text{ for all } x \in \mathbb{R}^n\bigr\}.$$

Put differently, $\|A\|$ is the minimal *Lipschitz constant* of the linear mapping $A\colon \mathbb{R}^n \to \mathbb{R}^n$.

Finally, we will shortly discuss the most important subordinate matrix norms:

- The 1-norm is induced by the 1-norm on $\mathbb{R}^n$, defined by

$$\|x\|_1 := \sum_{i=1}^n |x_i|.$$

The induced matrix norm has the form

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^n |a_{ij}|.$$

- The ∞-norm is induced by the ∞-norm on $\mathbb{R}^n$, defined by

$$\|x\|_\infty := \max_{1\le i\le n} |x_i|.$$

The induced matrix norm has the form

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^n |a_{ij}|.$$

- The 2-norm or *spectral norm* is induced by the Euclidean norm on $\mathbb{R}^n$, defined by

$$\|x\|_2 := \Bigl(\sum_{i=1}^n |a_{ij}|^2\Bigr)^{1/2}.$$

For the definition of the induced matrix norm, we have to recall that the *singular values* of a matrix $A \in \mathbb{R}^{n\times n}$ are precisely the square roots of the eigenvalues of the (symmetric and positive semi-definite) matrix A^TA. That is, $\sigma \ge 0$ is a singular value of A, if and only if σ^2 is an eigenvalue of A^TA. One then has

$$\|A\|_2 = \max\bigl\{\sigma \ge 0 : \sigma \text{ is a singular value of } A\bigr\}.$$

Remark 1. Another quite common matrix norm is the *Frobenius norm*

$$\|A\|_F := \Big(\sum_{i,j=1}^{n} |a_{ij}|^2\Big)^{1/2}.$$

One can show that this is indeed a matrix norm (that is, it is sub-multiplicative), but it is not induced by any norm on $\mathbb{R}^n$ unless $n = 1$. This can be easily seen by the fact that $\|\mathrm{Id}\|_F = \sqrt{n}$, which is different from 1 for $n > 1$.

3. Main Theorems

Recall that a *fixed point* of a mapping $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is a solution of the equation

$$\Phi(x) = x.$$

In other words, if we apply the mapping Φ to a fixed point, then we will obtain as a result the same point.

Theorem 2 (Banach's fixed-point theorem)**.** *Assume that $K \subset \mathbb{R}^n$ is closed and that $\Phi\colon K \to K$ is a contraction. That is, there exists $0 \le C < 1$ such that*

$$\|\Phi(x) - \Phi(y)\| \le C\|x - y\|$$

for all x, $y \in K$, where $\|\cdot\|$ is any norm on $\mathbb{R}^n$. Then the function Φ has a unique fixed point $\hat{x} \in K$.

Now let $x^{(0)} \in K$ be arbitrary and define the sequence $(x^{(k)})_{k\in\mathbb{N}} \subset K$ by $x^{(k+1)} := \Phi(x^{(k)})$. Then we have the estimates

$$\begin{aligned}
\|x^{(k+1)} - \hat{x}\| &\le C\|x^{(k)} - \hat{x}\|,\\
\|x^{(k)} - \hat{x}\| &\le \frac{C^k}{1-C}\|x^{(1)} - x^{(0)}\|,\\
\|x^{(k)} - \hat{x}\| &\le \frac{C}{1-C}\|x^{(k)} - x^{(k-1)}\|.
\end{aligned}$$

In particular, the seqeuence $(x^{(k)})_{k\in\mathbb{N}}$ converges to $\hat{x}$.

Proof. Assume that $\hat{x}$, $\hat{y}$ are two fixed points of Φ in K. Then

$$\|\hat{x} - \hat{y}\| = \|\Phi(\hat{x}) - \Phi(\hat{y})\| \le C\|\hat{x} - \hat{y}\|.$$

Wegen $0 \le C < 1$, it follows that $\hat{x} = \hat{y}$. Thus the fixed point (if it exists) is unique.

Let now $x^{(0)} \in K$ be arbitrary and define inductively $x^{(k+1)} = \Phi(x^{(k)})$. Then we have for all $k \in \mathbb{N}$ and $m \in \mathbb{N}$ the estimate

$$\begin{aligned}
\|x^{(k+m)} - x^{(k)}\| &\le \sum_{j=1}^{m} \|x^{(k+j)} - x^{(k+j-1)}\|\\
&\le \sum_{j=1}^{m} C^{j-1}\|x^{(k+1)} - x^{(k)}\|\\
&\le \sum_{j=1}^{\infty} C^{j-1}\|x^{(k+1)} - x^{(k)}\| \qquad (2)\\
&= \frac{1}{1-C}\|x^{(k+1)} - x^{(k)}\|\\
&\le \frac{C}{1-C}\|x^{(k)} - x^{(k-1)}\|\\
&\le \frac{C^k}{1-C}\|x^{(1)} - x^{(0)}\|.
\end{aligned}$$

This shows that the sequence $\{x^{(k)}\}_{k\in\mathbb{N}}$ is a Cauchy–sequence and thus has a limit $\hat{x} \in \mathbb{R}^n$. Since the set K is closed, we further obtain that $\hat{x} \in K$. Now note that the mapping Φ is by definition Lipschitz–continuous with Lipschitz–constant C. In particular, this shows that Φ is continuous. Thus

$$\hat{x} = \lim_{k\to\infty} x^{(k)} = \lim_{k\to\infty} x^{(k+1)} = \lim_{k\to\infty} \Phi(x^{(k)}) = \Phi\Bigl(\lim_{k\to\infty} x^{(k)}\Bigr) = \Phi(\hat{x}),$$

which shows that $\hat{x}$ is a fixed point of Φ. The different estimates claimed in the theorem now follow easily from the fact that Φ is a contraction and from (2) by considering the limit $m \to \infty$ (note that the norm is a continuous mapping, and thus the left hand side in said equation tends to $\|\hat{x} - x^{(k)}\|$). □

Theorem 3. *Assume that $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in a neighborhood of a fixed point $\hat{x}$ of Φ and that there exists a norm $\|\cdot\|$ on $\mathbb{R}^n$ with subordinate matrix norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$ such that*

$$\|D\Phi(\hat{x})\| < 1.$$

Then there exists a closed neighborhood K of $\hat{x}$ such that Φ is a contraction on K.

In particular, the fixed point iteration $x^{(k+1)} = \Phi(x^{(k)})$ converges for every $x^{(0)} \in K$ to $\hat{x}$.

Proof. Because of the continuous differentiability of Φ there exist $\rho > 0$ and a constant $0 < C < 1$ such that

$$\|D\Phi(x)\| \le C \qquad \text{for all } x \in B_\rho(\hat{x}) := \bigl\{y \in \mathbb{R}^n : \|y - \hat{x}\| \le \rho\bigr\}.$$

Let now x, $y \in B_\rho(\hat{x})$. Then

$$\Phi(y) - \Phi(x) = \int_0^1 D\Phi\bigl(x + t(y-x)\bigr)(y-x)\,dt.$$

Thus

$$\begin{aligned}\|\Phi(y) - \Phi(x)\| &\le \int_0^1 \|D\Phi\bigl(x + t(y-x)\bigr)(y-x)\|\,dt \\ &\le \int_0^1 \|D\Phi\bigl(x + t(y-x)\bigr)\|\|y-x\|\,dt \\ &\le C\|y-x\|.\end{aligned}$$

In particular, we obtain with $y = \hat{x}$ the inequality

$$\|\hat{x} - \Phi(x)\| = \|\Phi(\hat{x}) - \Phi(x)\| \le C\|\hat{x} - x\| \le C\rho < \rho.$$

Thus whenever $x \in B_\rho(\hat{x})$ we also have $\Phi(x) \in B_\rho(\hat{x})$. This shows that, indeed, Φ is a contraction on $B_\rho(\hat{x})$. Application of Banach's fixed point theorem yields the final claim of the theorem. □

Remark 4. Assume that $A \in \mathbb{R}^{n\times n}$ and define the *spectral radius*[1] of A by

$$\rho(A) := \max\bigl\{|\lambda| : \lambda \in \mathbb{C} \text{ is an eigenvalue of } A\bigr\}.$$

That is, $\rho(A)$ is (the absolute value of) the largest eigenvalue of A. Then every subordinate matrix norm on $\mathbb{R}^{n\times n}$ satisfies the inequality

$$\rho(A) \le \|A\|.$$

In case a largest eigenvalue is real, this easily follows from the fact that, whenever λ is an eigenvalue with eigenvector x, we have

$$|\lambda|\|x\| = \|\lambda x\| = \|Ax\| \le \|A\|\|x\|.$$

Division by $\|x\|$ yields the desired estimate.

[1] This is usually different from the spectral *norm* of A.

Conversely, it is also possible to show that for every $\varepsilon > 0$ there exists a norm $\|\cdot\|_\varepsilon$ on $\mathbb{R}^n$ such that the induced matrix norm $\|\cdot\|_\varepsilon$ on $\mathbb{R}^{n\times n}$ satisfies

$$\|A\|_\varepsilon \leq \rho(A) + \varepsilon$$

(this is a rather tedious construction involving real versions of the Jordan normal form of A).

Together, these results imply that there exists a subordinate matrix norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$ such that $\|A\| < 1$, if and only if $\rho(A) < 1$. Thus, the main condition of the previous theorem could also be equivalently reformulated as

$$\rho\big(D\Phi(\hat{x})\big) < 1.$$

4. Consequences for the Solution of Equations

Assume for simplicity that $n = 1$. That is, we are only trying to solve a single equation in one variable. The fixed point function will then have the form

$$\Phi(x) = x + g(x)f(x),$$

and g should be different from zero everywhere—or at least near the sought for solution.

Assume now that $f(\hat{x}) = 0$. Then Theorem 3 states that fixed point iteration will converge to $\hat{x}$ for all starting values $x^{(0)}$ sufficiently close to $\hat{x}$, if $|\Phi'(\hat{x})| < 1$ (note that on $\mathbb{R}$ there is only a single norm satisfying $|1| = 1$). Now, if f and g are sufficiently smooth, then the equation $f(\hat{x}) = 0$ implies that

$$\Phi'(\hat{x}) = 1 + g'(\hat{x})f(\hat{x}) + g(\hat{x})f'(\hat{x}) = 1 + g(\hat{x})f'(\hat{x}).$$

In particular, $|\Phi'(\hat{x})| < 1$, if and only if the following conditions hold:

(1) $f'(\hat{x}) \neq 0$.
(2) $\operatorname{sgn}(g(\hat{x})) = -\operatorname{sgn}(f'(\hat{x}))$.
(3) $|g(\hat{x})| < 2/|f'(\hat{x})|$.

This provides sufficient conditions for the convergence of the above fixed point iteration in one dimension.

In case of higher dimensions with fixed point function

$$\Phi(x) = x + G(x)F(x)$$

with $G\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$, we similarly obtain (though with more complicated calculations)

$$D\Phi(\hat{x}) = \operatorname{Id} + G(\hat{x}) \cdot DF(\hat{x}).$$

Consequently, said fixed point iteration will converge locally, if the matrix $DF(\hat{x})$ is invertible and the matrix $G(\hat{x})$ is sufficiently close to $-DF(\hat{x})^{-1}$.

5. The Theorey Behind Newton's Method

Newton's method (in one dimension) has the form of the fixed point iteration

$$x^{(k+1)} := \Phi(x^{(k)}) := x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}.$$

It is therefore of the form discussed above with

$$g(x) := -\frac{1}{f'(x)}.$$

In particular, this shows that it converges locally to a solution $\hat{x}$ of the equation $f(x) = 0$ provided that $f'(\hat{x}) \neq 0$. When actually performing the iterations, however, it turns out that the method works too well: While Banach's fixed point theorem predicts a linear convergence of the iterates to the limit $\hat{x}$, the actual convergence of the iterates is much faster.

Definition 5. We say that a sequence $(x^{(k)})_{k\in\mathbb{N}} \subset \mathbb{R}^n$ converges to $\hat{x}$ with *convergence order* (or: *convergence speed*) $p \geq 1$, if $x^{(k)} \to \hat{x}$ and there exists $0 < C < +\infty$ ($0 < C < 1$ in case $p = 1$) such that

$$\|x^{(k+1)} - \hat{x}\| \leq C\|x^{(k)} - \hat{x}\|^p.$$

In the particular case of $p = 1$ we speak of *linear convergence.* The rough interpretation is that in each step the number of "correct digits" increases by a constant number.

For $p = 2$ we speak of *quadratic convergence.* Here the number of "correct digits" roughly doubles in each step of the iteration.

In general, numerical methods with higher order are (all else being equal) preferable to methods with lower order, if one aims for (reasonably) high accuracy.

Theorem 6. *Assume that $\Phi\colon \mathbb{R} \to \mathbb{R}$ is p-times continuously differentiable with $p \geq 2$ in a neighborhood of a fixed point $\hat{x}$ of Φ. Assume moreover that*

$$0 = \Phi'(\hat{x}) = \Phi''(\hat{x}) = \ldots = \Phi^{(p-1)}(\hat{x}).$$

Then the fixed point sequence $x^{(k+1)} = \Phi(x^{(k)})$ converges to $\hat{x}$ with order (at least) p, provided that the starting point $x^{(0)}$ is sufficiently close to $\hat{x}$.

Proof. First note that Theorem 3 shows that the fixed point sequence indeed converges to $\hat{x}$ for suitable starting points $x^{(0)}$.

A Taylor expansion of Φ at the fixed point $\hat{x}$ now shows that

$$x^{(k+1)} - \hat{x} = \Phi(x^{(k)}) - \Phi(\hat{x}) = \sum_{s=1}^{p-1} \frac{\Phi^{(s)}(\hat{x})}{s!}(x^{(k)} - \hat{x})^s + \frac{\Phi^{(p)}(\xi)}{p!}(x^{(k)} - \hat{x})^p$$

for some ξ between $\hat{x}$ and $x^{(k)}$. Since $\Phi^{(s)}(\hat{x}) = 0$ for $0 \leq s \leq p-1$, this further implies that

$$|x^{(k+1)} - x^{(k)}| = \frac{|\Phi^{(p)}(\xi)|}{p!}|x^{(k)} - \hat{x}|^p.$$

Because $\Phi^{(p)}$ is continuous, there exists $C > 0$ such that

$$\frac{|\Phi^{(p)}(\xi)|}{p!} \leq C$$

for ξ sufficiently close to $\hat{x}$. Thus

$$|x^{(k+1)} - x^{(k)}| \leq C|x^{(k)} - \hat{x}|^p,$$

and therefore the convergence order of the sequence $(x^{(k)})_{k\in\mathbb{N}}$ is (at least) p. $\square$

Remark 7. It is possible to show conversely that the convergence order is *precisely* p, if in addition to the conditions of Theorem 6 the function Φ is $(p+1)$-times differentiable near $\hat{x}$ and $\Phi^{(p)}(\hat{x}) \neq 0$.

Corollary 8. *Newton's method converges at least quadratically provided that $f'(\hat{x}) \neq 0$ and f is twice differentiable.*

Remark 9. In higher dimensions, Newton's method is defined by the iteration

$$x^{(k+1)} = x^{(k)} + h^{(k)},$$

where $h^{(k)}$ solves the linear system

$$DF(x^{(k)})h = -F(x^{(k)}).$$

Equivalently,

$$x^{(k+1)} = x^{(k)} - DF(x^{(k)})^{-1}F(x^{(k)}).$$

Similarly as in the one-dimensional case, one can show that this method converges (at least) quadratically, provided that F is twice differentiable and $DF(\hat{x})$ is an invertible matrix.

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