NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **MA2501 Numerical Methods**

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Examination date: 07th June 2014

Examination time (from–to): 09:00–13:00

Permitted examination support material:

- The textbook: Cheney & Kincaid, Numerical Mathematics and Computing, 6. or 7. edition, including the list of errata.
- Rottmann, Mathematical formulae.
- Handouts on *Fixed point iterations* and *On the existence of a Cholesky factorization*.
- Approved basic calculator.

Other information:

- All answers should be justified and include enough details to make it clear which methods or results have been used.
- Some of the (sub-)problems will earn you more points than others — the total is 100 points.

Language: English

Number of pages: 2

Number pages enclosed: 0

Checked by:

Date Signature

Problem 1 Consider the linear system

$$\begin{pmatrix} 4 & 2 & 6 \\ 2 & 1 & 1 \\ -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 5 \end{pmatrix}.$$

a) Compute the solution of this system using Gaussian elimination with scaled partial pivoting.
(10 points)

b) Is it possible to solve this equation using Gaussian elimination without pivoting? Is it possible to apply Cholesky decomposition?
(5 points)

Problem 2 Consider the function

$$f(x) := 2x - \sin(x) + 2.$$

In order to solve the equation $f(x) = 0$, it is possible to apply a fixed point iteration of the form

$$x_{k+1} = x_k - \frac{1}{2} f(x_k).$$

a) Show that the equation $f(x) = 0$ has a unique solution $\hat{x}$, and that the iteration converges for every starting value $x_0 \in \mathbb{R}$ to $\hat{x}$.
(10 points)

b) Compute one step of the fixed point iteration with a starting value $x_0 = 0$. Use your result to estimate, after how many steps we have $|x_k - \hat{x}| \leq 2^{-20}$.
(5 points)

Problem 3 Denote by f_n, $n \in \mathbb{N}$, the polynomial of degree n that interpolates the function $f(x) = e^x + e^{-x}$ in equidistant interpolation points in the interval $[0, 1]$.

a) Show that $f_n(x) \to f(x)$ for every $x \in \mathbb{R}$.
(10 points)

b) Provide an estimate for

$$\sup_{0 \leq x \leq 1} |f_5(x) - f(x)|.$$

(10 points)

Problem 4 We are given a function $f\colon \mathbb{R} \to \mathbb{R}$ at the following points:

x_i	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
$f(x_i)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{2}$	2	4	16

Compute from these values the best possible approximation of $f'(0)$ using central finite differences and Richardson extrapolation.
(10 points)

Problem 5 Consider a quadrature rule of the form

$$Q(f,-1,1) := 2\Big(c_0 f(-1) + c_1 f(-2/3) + c_2 f(0) + c_3 f(2/3) + c_4 f(1)\Big)$$

for the approximation of a definite integral $\int_{-1}^{1} f(x)\,dx$.

a) Find weights $c_0,\dots,c_4 \in \mathbb{R}$ such that all polynomials of degree 4 are integrated exactly, that is,

$$Q(P,-1,1) = \int_{-1}^{1} P(x)\,dx$$

whenever P is a polynomial of degree 4.
(15 points)

b) Using the weights computed in the first part of the exercise, find the smallest integer $k \in \mathbb{N}$ for which $Q(x^k,-1,1) \neq \int_{-1}^{1} x^k\,dx$.
(5 points)

Problem 6 Consider the initial value problem

$$y' = \cos(y) - 2y,$$
$$y(0) = 0.$$

a) Apply two steps of the explicit Euler method[1] with a step size of $h = 1$ for the solution of this equation.
(5 points)

b) Apply two steps of the implicit Euler method[2] with a step size of $h = 1$ for the solution of this equation. In each step, use two steps of Newton's method (with a resonable starting value of your choice) for the solution of the non-linear equation you have to solve.
(15 points)

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[1] This method is simply called *Euler's method* in Cheney & Kincaid.
[2] Recall that this method is defined by the iteration $y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$.