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MA2501 Numerical Methods
Spring 2014

Exercise set 4

1 Use the bisection method for finding a solution of the equation $x^2 - 2 = 0$ (i.e., for computing $\sqrt{2}$) starting with the interval $[1, 2]$. Stop the computations when the error is smaller than 10^{-3}. How many iterations would be needed to obtain a result with an error smaller than 10^{-12}?

2 Implement the bisection method in Matlab. Your program should take as an input the two initial points $a < b$, the desired accuracy $\varepsilon > 0$, and a reference to the function f a root of which is to be computed.

Use your program for finding the solution of the equation $\tan(x) = 0$ in the interval $[2, 4]$ (which should be π) and a solution of the equation $3\cos(x)\sin(x^2) = 1$ in the interval $[0, 10]$. What happens if you apply the method to the equation $\tan(x) = 0$ with the starting interval $[1, 2]$?

3 Prove that the *regula falsi* method works. More precisely:

Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and that $a < b$ are such that $\operatorname{sgn}(f(a)) \neq \operatorname{sgn}(f(b))$. Denote by $[a_k, b_k]$ the interval one obtains after the k-th step of the *regula falsi* method.

a) Show that the sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ converge, respectively, to points $\hat{a} \leq \hat{b}$.

b) Show that at least one of the numbers $f(\hat{a})$ and $f(\hat{b})$ is equal to 0.

4 Assume that the *regula falsi* method (without modifications) is used for finding the solution of the equation $x^3 = 0$.

a) Show that during all the iterations one endpoint of the solution interval remains unchanged, unless the iteration finds the solution after the first step.

b) Denote by c_k the result of the method at the k-th step. Show that

$$\lim_{k\to\infty} \frac{|c_{k+1}|}{|c_k|} = 1. \tag{1}$$

Remark: A sequence $(c_k)_{k\in\mathbb{N}}$ converging to 0 and satisfying (1) is said to converge *sublinearly*. Sublinear converge of a sequence means that the number of iterations it takes to come closer to the limit by even one digit will become arbitrarily large. Thus numerical methods that have such sequences as output should usually be avoided.

$\boxed{5}$ Find an example of a function $\Phi\colon \mathbb{R} \to \mathbb{R}$ satisfying $|\Phi(x) - \Phi(y)| < |x - y|$ for all $x \neq y \in \mathbb{R}$, but which has no fixed point. Why is this example no contradiction to Banach's fixed point theorem? What happens if you apply fixed point iteration with this function Φ?

$\boxed{6}$ Apply fixed point iteration to the solution of the equation $\cos(x) = \frac{1}{2}\sin(x)$ using the mapping

$$\Phi(x) = x + \cos(x) - \frac{1}{2}\sin(x).$$

a) Show that the mapping Φ is a contraction on $[0, \pi/2]$ (note that you also have to show that $\Phi(x) \in [0, \pi/2]$ for all $x \in [0, \pi/2]$).

b) Compute the first five iterates of the fixed point iteration using the starting value $x^{(0)} = 0$.

c) Provide an estimate of the accuracy of the outcome of the method after the 5^{th} iteration. How many iterations will be needed to obtain a result with an error smaller than 10^{-12}?