Exam for MA2401 and MA6401, 4.5.2020

Group 1, Neutral Geometry

a) A Saccheri quadrilateral is a parallelogram. (true)

b) In a parallelogram, opposite sides are equally long. (false)

c) A Lambert quadrilateral cannot be a Saccheri quadrilateral. (false)

d) A Saccheri quadrilateral can be a Lambert quadrilateral. (true)

e) A Lambert quadrilateral is a parallelogram. (true)

Group 2, Neutral Geometry

a) Any triangle has an inscribed circle. (true)

b) No triangle has a circumscribed circle. (false)

c) A triangle has angle sum $180°$ if and only if it has a circumscribed circle. (false)

d) If there exists a triangle with angle sum $180°$ then all triangles have a circumscribed circle. (true)

e) In some geometries, there are triangles with angle sum $180°$ and there are triangles with angle sum less than $180°$.(false)

Group 3, Euclidian Geometry

a) The centroid of a triangle always lies inside the triangle. (true)

b) The orthocentre of a triangle always lies inside the triangle.(false)

c) The centroid and the orthocentre coincide if and only if the triangle is equilateral.(true)

d) The circumcentre, the centroid and the orthocentre are colinear.(true)

e) The circumcentre, the centroid and the orthocentre are colinear if and only if the triangle is equilateral.(false)

Group 4, Hyperbolic Geometry

a) A Lambert quadrilateral can be a Saccheri quadrilateral. (false)

b) A Saccheri quadrilateral cannot be a Lambert quadrilateral. (true)

c) A Saccheri quadrilateral is a parallelogram. (true)

d) Opposite sides in a Saccheri quadrilateral has a common perpendicular. (true)

e) A Lambert quadrilateral is a parallelogram. (true)

1 Neutral Geometry (2+1+2=5 points)

Define the distance between two points (x_1, y_1) and $(x_2, y_2) \in \mathbb{R}^2$ by

$$D((x_1, y_1), (x_2, y_2)) = max\{|x_2 - x_1|, |y_2 - y_1|\}.$$

(hereby, $max\{a, b\}$ denotes the larger of two real numbers a and b)

a) Verify that D defines a metric.

b) Find all points (x, y) in $\mathbb{R}^2$ such that $D((0, 0), (x.y)) = 1$. Draw a sketch in the Cartesian plane. (This should explain the name *square metric*).

c) Let l be a line defined by the equation $y = m \cdot x + b$, with $m, b \in \mathbb{R}$. Show that for $|m| \geq 1$, the function $f : l \to \mathbb{R}$, $f(x, y) = m \cdot x$ defines a coordinate function, using the square metric.

Possible Solution:
Let $P = (x_1, y_1), Q = (x_2, y_2)$ be two arbitrary points.

a) We have to prove the following properties of a metric (see 3.2.9):

1. $D(P, Q) = D(Q, P)$ for every P and Q.
Proof:

$$\begin{aligned} D(P, Q) = D((x_1, y_1), (x_2, y_2)) =& max\{|x_2 - x_1|, |y_2 - y_1|\} \\ =& max\{|-(x_1 - x_2)|, |-(y_1 - y_2)|\} \\ =& max\{|x_1 - x_2|, |y_1 - y_2|\} \\ =& D((x_2, y_2), (x_1, y_1)) = D(Q, P). \end{aligned}$$

2. $D(P, Q) \geq 0$ for every P and Q.
Proof:

$$D(P, Q) = D((x_1, y_1), (x_2, y_2)) = max\{|x_2 - x_1|, |y_2 - y_1|\} \geq 0$$

since $|x_2 - x_1| \geq 0$ and $|y_2 - y_1| \geq 0$.

3. $D(P, Q) = 0$ if and only if $P = Q$.
Proof: We have to show two directions.
Assume first: $D(P, Q) = 0$.

$$\begin{aligned} &\Rightarrow max\{|x_2 - x_1|, |y_2 - y_1|\} = 0 \\ &\Rightarrow |x_2 - x_1| = 0 \text{ and } |y_2 - y_1| = 0 \\ &\Rightarrow x_2 = x_1 \text{ and } y_2 = y_1 \\ &\Rightarrow P = Q. \end{aligned}$$

Now assume $P = Q$.
$\Rightarrow x_1 = x_2$ and $y_1 = y_2$.
$\Rightarrow max\{|x_2 - x_1|, |y_2 - y_1|\} = max\{0, 0\} = 0$

b)

$$\begin{aligned} 1 = D((0, 0), (x, y)) =& max\{|0 - x|, |0 - y|\} \\ =& max\{|x|, |y|\} \end{aligned}$$

$\Rightarrow$ either $x = \pm 1$ and $-1 \leq y \leq 1$, or $-1 \leq x \leq 1$ and $y = \pm 1$.

c) We have to prove two things (see 3.2.13)

- f is a one-to-one correspondence between l and $\mathbb{R}$. This follows from the fact that f is a linear function, and therefore one-to-one and onto (cf. E.4 in the book).
- $PQ = |f(P) - f(Q)|$.
For the two points P and Q both lying on the line l, we have $P = (x_1, m \cdot x_1 + b)$ and $Q = (x_2, m \cdot x_2 + b)$. If follows

$$\begin{aligned} PQ =& max\{|x_2 - x_1|, |m \cdot x_2 + b - (m \cdot x_1 + b)|\} \\ =& max\{|x_2 - x_1|, |m \cdot (x_2 - x_1)|\} \\ \overset{m \text{ positive}}{=}& max\{|x_2 - x_1|, m \cdot |x_2 - x_1|\} \end{aligned}$$

Now since $m \geq 1$, we have $|x_2 - x_1| \leq m \cdot |x_2 - x_1|$, so it follows for the above

$$=m \cdot |x_2 - x_1| = |m \cdot x_2 - m \cdot x_1| = |f(P) - f(Q)|.$$

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2 Euclidian Geometry (2+2+2+2=8 points)

An equilateral triangle is one which all three sides have equal length.

a) Prove that a Euclidian triangle is equilateral if and only if each of its angles measures 60°.

b) Prove that there is an equilateral triangle in Euclidian geometry.

c) Split an equilateral triangle at the midpoint of one side to prove that there is a triangle whose angles measure $30^\circ, 60^\circ$ and 90°.

d) Prove that, in any triangle with angle measures $30^\circ, 60^\circ$ and 90°, the length of the side opposite the 30° angle is one half the length of the hypotenuse.

Possible Solution:

a) Proof:

„⇒“ We assume that the triangle $\triangle ABC$ is equilateral, so $\overline{AB} \cong \overline{AC} \cong \overline{BC}$ (hyp.).
We can apply the Isosceles Triangle Theorem (3.6.5) for each pair of sides (e.g. we obtain $\angle ABC \cong \angle ACB$ since $\overline{AB} \cong \overline{AC}$, and $\angle ACB \cong \angle CAB$ since $\overline{AC} \cong \overline{BC}$, etc.). Therefore, we get

$$\angle ABC \cong \angle ACB \cong \angle CAB.$$

From the Angle Sum Theorem (5.1.3) in Euclidian Geometry, we have that

$$\begin{aligned} 180^\circ =& \mu(\angle ABC) + \mu(\angle ACB) + \mu(\angle CAB) = 3 \cdot \mu(\angle ABC) \\ \Leftrightarrow 60^\circ =& \mu(\angle ABC) = \mu(\angle ACB) = \mu(\angle CAB). \end{aligned}$$

„⇐“ Now we assume $\angle ABC \cong \angle ACB \cong \angle CAB$ and $60^\circ = \mu(\angle ABC) = \mu(\angle ACB) = \mu(\angle CAB)$ (hyp.)
By the Converse of the Isosceles Triangle Theorem (4.2.2), it follows that $\triangle ABC$ is isosceles with $\overline{AB} \cong \overline{AC}$ since $\angle ABC \cong \angle ACB$. Similarly, we obtain $\overline{AC} \cong \overline{BC}$ and $\overline{BC} \cong \overline{AB}$. Together, we get $\overline{AB} \cong \overline{AC} \cong \overline{BC}$, so $\triangle ABC$ is an equilateral triangle.

b) Proof:
We construct an equilateral triangle. Let A and B be two distinct points. By the protractor postulate, we can construct a point A' such that $\mu(\angle BAA') = 60^\circ$. We can also construct a point B' on the same side of $\overleftrightarrow{AB}$ such that $\mu(\angle ABB') = 60^\circ$ (protractor posulate). Since $\mu(\angle BAA') + \mu(\angle ABB') = 60^\circ + 60^\circ = 120^\circ < 180^\circ$, it follows from Euclid's Postulate V that the lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$ intersect on one side of $\overleftrightarrow{AB}$. We call the intersection point C. The three points A, B and C are noncollinear by construction, so there is a triangle $\triangle ABC$ with angle sum 180° (Angle Sum Theorem 5.1.3 in Euclidian Geometry). So we deduce:

$$\begin{aligned} 180^\circ =& \mu(\angle CAB) + \mu(\angle ABC) + \mu(\angle ACB) = 60^\circ + 60^\circ + \mu(\angle ACB) \\ \Rightarrow 60^\circ =& \mu(\angle ACB). \end{aligned}$$

So we have constructed a triangle $\triangle ABC$ with three angles that measure 60° each, which is (by part a)) an equilateral triangle.

c) Proof:
We assume that $\triangle ABC$ is an equilateral triangle (that exists by part b)). So we have $60^\circ = \mu(\angle ABC) = \mu(\angle ACB) = \mu(\angle CAB)$.
We construct a triangle whose angles measure $30^\circ, 60^\circ$ and 90°.

Let M be the midpoint of $\overline{AB}$ (that exists by the ruler postulate). Now by SSS, we have $\triangle AMC \cong \triangle BMC$ (since $\overline{AM} \cong \overline{BM}, \overline{AC} \cong \overline{BC}, \overline{MC} \cong \overline{MC}$). So we have

$$\begin{aligned} & \angle ACM \cong \angle BCM \\ \text{(protractor postulate, part 4)} \Rightarrow 60^\circ =& \mu(\angle ACB) = \mu(\angle ACM) + \mu(\angle BCM) = 2 \cdot \mu(\angle ACM) \\ \Leftrightarrow 30^\circ =& \mu(\angle ACM). \end{aligned}$$

Finally, an application of the Angle Sum Theorem (5.1.3) leads to

$$\begin{aligned} 180^\circ =& \sigma(\triangle AMC) = \mu(\angle CAM) + \mu(\angle ACM) + \mu(\angle CMA) = 60^\circ + 30^\circ + \mu(\angle CMA) \\ \Leftrightarrow 90^\circ =& \mu(\angle CMA). \end{aligned}$$

So we have constructed a triangle $\triangle ACM$ whose angles measure $30^\circ, 60^\circ$ and 90°.

d) Proof:
Let $\triangle ABC$ be a triangle with a right angle at C, i.e. $\mu(\angle ACB) = 90^\circ$, and let $\mu(\angle CAB) = 30^\circ$ and $\mu(\angle ABC) = 60^\circ$.
We prove: $\frac{1}{2} \cdot AB = BC$.

Let M be the midpoint of $\overline{AB}$. Then, by Theorem 8.3.3, it follows $AM = MC$. So $\triangle MCB$ is an isosceles triangle (with $\overline{AM} \cong \overline{MB} \cong \overline{MC}$. So by the Isosceles Triangle Theorem (3.6.5), it follows $\angle CBM \cong \angle MCB$. By assumption, we have $\mu(\angle CBM) = 60^\circ$, so $\mu(\angle CMB) = 60^\circ$, and by the Angle Sum Theorem (5.1.3) it follows also $\mu(\angle CMB) = 60^\circ$. So by part a), we can conclude that $\triangle BCM$ is an equilateral triangle, and therefore $\overline{BC} \cong \overline{MB}$. Since M was de midpoint of $\overline{AB}$, it follows $BC = \frac{1}{2} \cdot AB$.

3 Hyperbolic Geometry (3+3=6 points)

Let $\triangle ABC$ be a triangle and let D, E and F be the midpoints of the sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively.

a) Prove that $\triangle EDC$ is not similar to $\triangle ABC$.

b) Prove that the congruences $\overline{AF} \cong \overline{ED}$, $\overline{AE} \cong \overline{FD}$ and $\overline{BD} \cong \overline{EF}$ cannot all hold.

Possible Solution:

a) In hyperbolic geometry, similar triangles are also congruent (Theorem 6.1.11, AAA). So we prove here: $\triangle EDC \ncong \triangle ABC$.
Proof:
Without loss of generality, we assume $AC \leq BC$ (if necessary, we can re-name the vertices of the triangle such that this holds).
Since E is the midpoint of $\overline{AC}$ and D is the midpoint of $\overline{BC}$, it follows that $EC \leq DC(*)$.
Assume now that $\triangle EDC \cong \triangle ABC$ (RAA hypothesis). Then either $\overline{CE} \cong \overline{CA}$ or $\overline{CE} \cong \overline{CB}$, since $\angle ECD \cong \angle ACB$.
Since E is the midpoint of $\overline{AC}$, we have $CE < CA$. So it must hold $\overline{CE} \cong \overline{CB} \Rightarrow CE = CB$. Since D is the midpoint of $\overline{CB}$, we have $CD < CB$. Together with $CE \leq DC$ (from $(*)$ above), we get a contradiction.
So we must decline the RAA hypothesis $\triangle EDC \cong \triangle ABC$ and get $\triangle EDC \ncong \triangle ABC$.

b) Proof:
Let $\triangle ABC$ be a triangle as in part a).
We assume $\overline{AF} \cong \overline{ED}$, $\overline{AE} \cong \overline{FD}$ and $\overline{BD} \cong \overline{EF}$ (RAA hypothesis).
By construction of the midpoints D, E and F, we have

$$\begin{aligned} &\overline{AE} \cong \overline{EC}, \overline{BD} \cong \overline{DC} \text{ and } \overline{AF} \cong \overline{FB} \\ \Rightarrow &\triangle EAF \cong \triangle EDC \text{ (SSS)} \\ \Rightarrow &\angle CED \cong \angle AEF \cong \angle CAB \text{ (*)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} &\triangle BDF \cong \triangle DCE \text{ (SSS for } \overline{BD} \cong \overline{DC}, \overline{DF} \cong \overline{CE} \text{ and } \overline{FB} \cong \overline{ED}) \\ \Rightarrow &\angle EDC \cong \angle FBD \cong \angle ABC \text{ (**)}. \end{aligned}$$

From $(*)$ and $(**)$ together, we have $\angle EDC \cong \angle ABC$, $\angle CED \cong \angle CAB$, and together with $\angle DCE \cong \angle ACB$ (which holds by construction of the two triangles), we obtain

$$\triangle EDC \cong \triangle ABC,$$

which is a contradiction to part a). So we must decline the RAA hypothesis and deduce that the three congruences $\overline{AF} \cong \overline{ED}$, $\overline{AE} \cong \overline{FD}$ and $\overline{BD} \cong \overline{EF}$ cannot all hold.

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4 Neutral Geometry (3+3=6 points)

a) Let a and c be two numbers such that $0 < a < c$. Prove that there exists a triangle $\triangle ABC$ such that $\angle BCA$ is a right angle, $BC = a$, and $AB = c$.

b) Let γ be a circle and let P be a point outside γ: Prove that there exist two lines through P that are tangent to γ.

Possible Solution:

a) Proof:
We start with arbitrary, but fixed numbers a, c such that $0 < a < c$.

Let B be a point. Using the ruler postulate (for any line that is incident with B), we can find a point C such that $BC = a$. We construct a perpendicular to the line $\overleftrightarrow{BC}$ through the point C, and call it k (so we have $k \perp \overleftrightarrow{BC}$, $C \in k$). Now we consider the circle γ with radius c and center B. Since $BC = a < c$, the point C is inside the circle γ. By Theorem 8.1.12, any line through C intersects γ, in particular also the line k. The intersection point of γ and k we call A.
So since $A \in \gamma$ by construction, we have $AB = c$ (= radius of γ). By construction, we have $\mu(\angle ACB) = 90^\circ$ and $BC = a$. So we have constructed a right triangle $\triangle ABC$ with right angle $\angle BCA$, $BC = a$, and $AB = c$.

b) Proof:
Let $\gamma = \mathcal{C}(O, r)$ be a circle and P be a point outside γ, so $OP > r$.
We apply now part a) of for $a = r > 0$ and $c = OP > r$. Following from a), there is a right triangle $\triangle OPC$ with right angle at C and $OC = a = r$, so C is a point on γ.
Now since $OC = r$ and $\overleftrightarrow{OC} \perp \overleftrightarrow{CP}$ (since $\mu(\angle OCP) = 90^\circ$), the line $\overleftrightarrow{CP}$ is tangent to γ (in C) (Application of the Tangent Line Theorem, 8.1.7).
Now, we drop a perpendicular from C to $\overleftrightarrow{OP}$ and call the foot Q. Let C' be the point on $\overleftrightarrow{CQ}$ such that $C * Q * C'$ and $\overline{CQ} \cong \overline{QC'}$. Now by SAS, we have $\triangle OCQ \cong \triangle OC'Q$ ($\angle CQO \cong \angle C'QO$ both right angles, $\overline{CQ} \cong \overline{QC'}$ by construction, and $\overline{OQ} \cong \overline{OQ}$).
So it follows that $\overline{OC} \cong \overline{OC'} = r$, so C' is on the circle γ. Since C' is by construction on the opposite side of $\overleftrightarrow{OP}$ than C, we have $C \neq C'$, so we indeed found two distinct points.
Since $\triangle OCQ \cong \triangle OC'Q$, we also have $\angle POC = \angle QOC \cong \angle QOC' = \angle POC'$, and therefore also $\triangle OCP \cong \triangle OC'P$ (SAS for $\angle POC \cong \angle POC'$, $\overline{OC} \cong \overline{OC'}$, and $\overline{OP} \cong \overline{OP}$). So in particular, we get $\angle PC'O \cong \angle PCO$, and $90^\circ = \mu(\angle PCO) = \mu(\angle PC'O)$, we get that the triangle $\triangle PC'O$ has a right angle at C', and therefore $\overleftrightarrow{C'P} \perp \overline{OC'}$, which means (again by the Tangent Line Theorem 8.1.7) that the line $\overleftrightarrow{C'P}$ is the second tangent line to the circle γ through the point P.