Exam for MA2401 and MA6401, 4.5.2020

True or false statements

Decide if the following statements are **true** or **false** in the indicated geometry. You get
+1 point for a correct answer,
-1 for an incorrect answer, and
0 points if you do not choose one of the alternatives (**no decision**).
For one group of statements, you can get at least 0 points in total (so you cannot get negative points for a task).

Group 1, Neutral Geometry

a) A Saccheri quadrilateral is a parallelogram.

b) In a parallelogram, opposite sides are equally long.

c) A Lambert quadrilateral cannot be a Saccheri quadrilateral.

d) A Saccheri quadrilateral can be a Lambert quadrilateral.

e) A Lambert quadrilateral is a parallelogram.

Group 2, Neutral Geometry

a) Any triangle has an inscribed circle.

b) No triangle has a circumscribed circle.

c) A triangle has angle sum 180° if and only if it has a circumscribed circle.

d) If there exists a triangle with angle sum 180° then all triangles have a circumscribed circle.

e) In some geometries, there are triangles with angle sum 180° and there are triangles with angle sum less than 180°.

Group 3, Euclidian Geometry

a) The centroid of a triangle always lies inside the triangle.

b) The orthocentre of a triangle always lies inside the triangle.

c) The centroid and the orthocentre coincide if and only if the triangle is equilateral.

d) The circumcentre, the centroid and the orthocentre are colinear.

e) The circumcentre, the centroid and the orthocentre are colinear if and only if the triangle is equilateral.

Group 5, Hyperbolic Geometry

a) A Lambert quadrilateral can be a Saccheri quadrilateral.

b) A Saccheri quadrilateral cannot be a Lambert quadrilateral.

c) A Saccheri quadrilateral is a parallelogram.

d) Opposite sides in a Saccheri quadrilateral has a common perpendicular.

e) A Lambert quadrilateral is a parallelogram.

1 Neutral Geometry (2+1+2=5 points)

Define the distance between two points (x_1, y_1) and $(x_2, y_2) \in \mathbb{R}^2$ by

$$D((x_1, y_1), (x_2, y_2)) = max\{|x_2 - x_1|, |y_2 - y_1|\}.$$

(hereby, $max\{a, b\}$ denotes the larger of two real numbers a and b)

a) Verify that D defines a metric.

b) Find all points (x, y) in $\mathbb{R}^2$ such that $D((0, 0), (x.y)) = 1$. Draw a sketch in the Cartesian plane. (This should explain the name *square metric*).

c) Let l be a line defined by the equation $y = m \cdot x + b$, with $m, b \in \mathbb{R}$. Show that for $|m| \geq 1$, the function $f : l \to \mathbb{R}$, $f(x, y) = m \cdot x$ defines a coordinate function, using the square metric.

2 Euclidian Geometry (2+2+2+2=8 points)

An equilateral triangle is one which all three sides have equal length.

a) Prove that a Euclidian triangle is equilateral if and only if each of its angles measures $60°$.

b) Prove that there is an equilateral triangle in Euclidian geometry.

c) Split an equilateral triangle at the midpoint of one side to prove that there is a triangle whose angles measure $30°, 60°$ and $90°$.

d) Prove that, in any triangle with angle measures $30°, 60°$ and $90°$, the length of the side opposite the $30°$ angle is one half the length of the hypotenuse.

3 Hyperbolic Geometry (3+3=6 points)

Let $\triangle ABC$ be a triangle and let D, E and F be the midpoints of the sides $\overline{BC}, \overline{AC}$, and $\overline{AB}$, respectively.

a) Prove that $\triangle EDC$ is not similar to $\triangle ABC$.

b) Prove that the congruences $\overline{AF} \cong \overline{ED}$, $\overline{AE} \cong \overline{FD}$ and $\overline{BD} \cong \overline{EF}$ cannot all hold.

4 Neutral Geometry (3+3=6 points)

a) Let a and c be two numbers such that $0 < a < c$. Prove that there exists a triangle $\triangle ABC$ such that $\angle BCA$ is a right angle, $BC = a$, and $AB = c$.

b) Let γ be a circle and let P be a point outside γ: Prove that there exist two lines through P that are tangent to γ.