Rotations: A different approach

Harald Hanche-Olsen

April 10, 2016

The aim of this little note is to give a treatment of rotations different from that given in chapter 10 of Gerard A. Venema: *Foundations of Geometry*, second edition (which I will simply refer to as “the book” from now on). Our setting is that of Chapter 10 of the book, on transformations. Any references such as to Lemma 10.1.10 are to the book; references within this note are just a single number.

We start with a simple lemma that may be seen as a precursor to Lemma 10.1.10. We leave its simple proof to the reader:

Lemma 1. *Any isometry that fixes two distinct points A and B is either the identity or the reflection through the line $\overleftrightarrow{AB}$.*

Definition 2. A *rotation* is an isometry with exactly one fixed point. A *rotation about O* is a rotation that fixes O.

Some comments: The book defines rotations differently, by a specific construction. We shall see that, in the end, our definition departs from the book’s definition in two respects: First, the book considers the identity to be a rotation (generated by the collapsed angle $\angle AOB$, where $\overrightarrow{OA} = \overrightarrow{OB}$), while we don’t. Second, the book gives a special name *half-turn* to a 180° rotation, probably because the construction of rotations does not accomodate half-turns, as 180° angles don’t exist in the book’s universe.

Be that as it may, our first result is our version of part 2 of Theorem 10.2.5.

Theorem 3. *Let R be a rotation about O. If n is any line through O (i.e., $O \in n$), then there exist lines s and t so that $R = \rho_s \circ \rho_n = \rho_n \circ \rho_t$. Conversely, if s and n are distinct lines through O, then $\rho_s \circ \rho_n$ is a rotation about O.*

Proof. Pick a point $P \in n$, $P \neq O$, and let $P' = R(P)$. Then by assumption (R has no fixed points other than O), $P' \neq P$. Let s be the perpendicular bisector of $\overline{PP'}$. Then $\rho_s \circ R(P) = \rho_s(P') = P$, so P is a fixed point of $\rho_s \circ R$. We also find $OP' = R(O)R(P) = OP$, so that $O \in s$, and hence O is another fixed point of $\rho_s \circ R$. Since $n = \overleftrightarrow{OP}$, Lemma 1 implies that either $\rho_s \circ R = \iota$ or $\rho_s \circ R = \rho_n$. Composing with ρ_s on the left, we conclude that either $R = \rho_s$ or $R = \rho_s \circ \rho_n$. But the former contradicts the assumption because ρ_s has many fixed points, not just one. So $R = \rho_s \circ \rho_n$.

We pause to note that this implies that R is invertible, since reflections are invertible, and any composition of invertible transformations is invertible.

Furthermore, R^{-1} is also a rotation about O, and so by the first part of the proof there exists a line t though O so that $R^{-1} = \rho_t \circ \rho_n$. Taking inverses, noting that any reflection is its own inverse, we conclude that $R = \rho_n \circ \rho_t$.

For the converse, take distinct lines s and n through O, and let $R = \rho_s \circ \rho_n$. Clearly, O is a fixed point of R. Assume P is a fixed point of R, i.e., $\rho_s\big(\rho_n(P)\big) = P$. Applying ρ_s on both sides, we conclude $\rho_n(P) = \rho_s(P)$. Put $Q = \rho_n(P)$. If $Q \neq P$ then n is the perpendicular bisector of $\overline{PQ}$, and so is s (since $\rho_s(P) = Q$). But n and s were assumed to be different lines, so this is a contradiction, and we must have $Q = P$. But then $P = Q = \rho_n(P)$ implies $P \in n$, and $P = Q = \rho_s(P)$ imples $P \in s$, so we must have $P = O$. In other words, O is the only fixed point of R, so R is a rotation about O. $\square$

We leave it as an exercise for the reader to verify that the rotation R satisfies $P * O * P'$ for at least one $P \neq O$, and equivalently for all $P \neq O$, if and only if $n \perp t$. This is the *half-turn* case.

Next, we tackle Lemma 10.3.3, but we tack on an important corollary.

Lemma 4. *If ℓ, m, and n are three lines (distinct or not) through a common point O, there exists a line s through O such that $\rho_\ell \circ \rho_m \circ \rho_n = \rho_s$. In particular, $\rho_\ell \circ \rho_m \circ \rho_n = \rho_n \circ \rho_m \circ \rho_\ell$.*

Proof. If $\ell = m$, there is nothing to prove; just take $s = n$. Otherwise, $\rho_\ell \circ \rho_m$ is a rotation, and hence Theorem 3 implies the existence of a line s so that $\rho_\ell \circ \rho_m = \rho_n \circ \rho_s$. Composing with ρ_s on the right, we get $\rho_\ell \circ \rho_m \circ \rho_n = \rho_s$, as claimed.

Now observe that ρ_s is its own inverse, and hence $\rho_\ell \circ \rho_m \circ \rho_n$ is its own inverse. But that inverse is $\rho_n \circ \rho_m \circ \rho_\ell$, and so the final equality is established. $\square$

We immediately conclude that any two rotations about O commute:

Corollary 5. *If R and S are two rotations about O, then $R \circ S = S \circ R$.*

Proof. Write $R = \rho_\ell \circ \rho_m$ and $S = \rho_n \circ \rho_t$. Then

$$R \circ S = \overbrace{\rho_\ell \circ \rho_m \circ \rho_n}^{\text{A}} \circ \rho_t = \overbrace{\rho_n \circ \underbrace{\rho_m \circ \rho_\ell}_{\text{B}}} ^{\text{A}}\underbrace{{}\circ \rho_t}_{} = \rho_n \circ \underbrace{\rho_t \circ \rho_\ell \circ \rho_m}_{\text{B}} = S \circ R,$$

where we used the final formula from Lemma 4 on the subexpressions marked A and B. $\square$

We can now state and prove our version of part 1 of Theorem 10.2.5.

Theorem 6. *Let R be a rotation about O, and $P \neq O$, $Q \neq O$. Write $P' = R(P)$ and $Q' = R(Q)$. Then either both $P * O * P'$ and $Q * O * Q'$ hold, or neither holds. In the latter case, $\angle POP' \cong \angle QOQ'$.*

Proof. First, if $Q \in \overleftrightarrow{OP}$, the result is immediate. So we assume otherwise. Second, we can, without loss of generality, move Q along the ray $\overrightarrow{OQ}$ to make $OQ = OP$.

Then we can find a second rotation S so that $S(P) = Q$: Simply let m be the perpendicular bisector of $\overline{PQ}$, and note that $O \in m$. Further, let $n = \overleftrightarrow{OP}$, and let $S = \rho_m \circ \rho_n$. Then $S(P) = \rho_m(\rho_n(P)) = \rho_m(P) = Q$, as claimed.

Thanks to Corollary 5, we also get $S(P') = S\big(R(P)\big) = R\big(S(P)\big) = R(Q) = Q'$.

From the three equations $S(O) = O$, $S(P) = Q$, and $S(P') = Q'$, plus the fact that S is an isometry and hence preserves colinearity, the relation $\cdot * \cdot * \cdot$, and angles, the conclusion readily follows. $\square$