Isometries: A different approach

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The main aim of this little note is to give a treatment of rotations different from that given in chapter 10 of Gerard A. Venema: *Foundations of Geometry*, second edition (which I will simply refer to as "the book" from now on). I also append a short section treating translations and glide reflections similarly. Here, the deviation from the book is much less dramatic, however. Our setting is that of Chapter 10 of the book, on transformations. Any references such as to Lemma 10.1.10 are to the book; references within this note are just a single number.

We assume the basic properties of isometries known: That they preserve colinearity, angle measure, and so on.

Here is a simple lemma that may be seen as a precursor to Lemma 10.1.10. We leave its simple proof to the reader:

Lemma 1. *Any isometry that fixes two distinct points A and B is either the identity or the reflection through the line $\overleftrightarrow{AB}$.*

Definition 2. A *rotation about O* (or with *center O*) is an isometry with fixed point O and no other fixed points. A *rotation* is a rotation about some point.

Some comments: The book defines rotations concretely, by a specific construction. We shall see that, in the end, our definition departs from the book's definition in two respects: First, the book considers the identity to be a rotation (generated by the collapsed angle $\angle AOB$, where $\overrightarrow{OA} = \overrightarrow{OB}$), while we don't. Second, the book gives a special name *half-turn* to a $180°$ rotation, probably because the construction of rotations does not accomodate half-turns, as $180°$ angles don't exist in the book's universe.

Be that as it may, here is our version of part 2 of Theorem 10.2.5.

Theorem 3. *Let R be a rotation about O. If n is any line through O (i.e., $O \in n$), then there exist lines s and t so that $R = \rho_s \circ \rho_n = \rho_n \circ \rho_t$. Conversely, if s and n are distinct lines through O, then $\rho_s \circ \rho_n$ is a rotation about O.*

Proof. Pick a point $P \in n$, $P \neq O$, and let $P' = R(P)$. Then by assumption (R has no fixed points other than O), $P' \neq P$. Let s be the perpendicular bisector of $\overline{PP'}$. Then $\rho_s\big(R(P)\big) = \rho_s(P') = P$, so P is a fixed point of $\rho_s \circ R$. We also find $OP' = R(O)R(P) = OP$, so that $O \in s$, and hence O is another fixed point of $\rho_s \circ R$. Since $n = \overleftrightarrow{OP}$, Lemma 1 implies that either $\rho_s \circ R = \iota$ or $\rho_s \circ R = \rho_n$. Composing with ρ_s on the left, we conclude that either $R = \rho_s$ or $R = \rho_s \circ \rho_n$. But the former contradicts the assumption because ρ_s has many fixed points, not just one. So $R = \rho_s \circ \rho_n$.

We pause to note that this implies that R is invertible, since reflections are invertible, and any composition of invertible transformations is invertible.

Furthermore, R^{-1} is also a rotation about O, and so by the first part of the proof there exists a line t though O so that $R^{-1} = \rho_t \circ \rho_n$. Taking inverses, noting that any reflection is its own inverse, we conclude that $R = \rho_n \circ \rho_t$.

For the converse, take distinct lines s and n through O, and let $R = \rho_s \circ \rho_n$. Clearly, O is a fixed point of R. Assume P is a fixed point of R, i.e., $\rho_s\big(\rho_n(P)\big) = P$. Applying ρ_s on both sides, we conclude $\rho_n(P) = \rho_s(P)$. Put $Q = \rho_n(P)$. If $Q \neq P$ then n is the perpendicular bisector of $\overline{PQ}$, and so is s (since $\rho_s(P) = Q$). But n and s were assumed to be different lines, so this is a contradiction, and we must have $Q = P$. But then $P = Q = \rho_n(P)$ implies $P \in n$, and $P = Q = \rho_s(P)$ imples $P \in s$, so we must have $P = O$. In other words, O is the only fixed point of R, so R is a rotation about O. □

We leave it as an exercise for the reader to verify that the rotation R satisfies $P * O * P'$ for at least one $P \neq O$, and equivalently for all $P \neq O$, if and only if $n \perp t$. This is the *half-turn* case.

Next, we tackle Lemma 10.3.3, but we tack on an important corollary.

Lemma 4. *If ℓ, m, and n are three lines (distinct or not) through a common point O, there exists a line s through O such that $\rho_\ell \circ \rho_m \circ \rho_n = \rho_s$. In particular, $\rho_\ell \circ \rho_m \circ \rho_n = \rho_n \circ \rho_m \circ \rho_\ell$.*

Proof. If $\ell = m$, there is nothing to prove; just take $s = n$. Otherwise, $\rho_\ell \circ \rho_m$ is a rotation, and hence Theorem 3 implies the existence of a line s so that $\rho_\ell \circ \rho_m = \rho_n \circ \rho_s$. Composing with ρ_s on the right, we get $\rho_\ell \circ \rho_m \circ \rho_n = \rho_s$, as claimed.

Now observe that ρ_s is its own inverse, and hence $\rho_\ell \circ \rho_m \circ \rho_n$ is its own inverse. But that inverse is $\rho_n \circ \rho_m \circ \rho_\ell$, and so the final equality is established. □

We immediately conclude that any two rotations about O commute:

Corollary 5. *If R and S are two rotations about O, then $R \circ S = S \circ R$.*

Proof. Write $R = \rho_\ell \circ \rho_m$ and $S = \rho_n \circ \rho_t$. Then

$$R \circ S = \overbrace{\rho_\ell \circ \rho_m \circ \rho_n}^{\text{A}} \circ \rho_t = \overbrace{\rho_n \circ \underbrace{\rho_m \circ \rho_\ell}_{\text{B}}}^{\text{A}}\!\!\underbrace{{}\circ \rho_t}_{} = \rho_n \circ \underbrace{\rho_t \circ \rho_\ell \circ \rho_m}_{\text{B}} = S \circ R,$$

where we used the final formula from Lemma 4 on the subexpressions marked A and B. □

We can now state and prove our version of part 1 of Theorem 10.2.5.

Theorem 6. *Let R be a rotation about O, and $P \neq O$, $Q \neq O$. Write $P' = R(P)$ and $Q' = R(Q)$. Then either both $P * O * P'$ and $Q * O * Q'$ hold, or neither holds. In the latter case, $\angle POP' \cong \angle QOQ'$.*

Proof. First, if $Q \in \overleftrightarrow{OP}$, the result is immediate. So we assume otherwise. Second, we can, without loss of generality, move Q along the ray $\overrightarrow{OQ}$ to make $OQ = OP$.

Then we can find a second rotation S so that $S(P) = Q$: Simply let m be the perpendicular bisector of $\overline{PQ}$, and note that $O \in m$. Further, let $n = \overleftrightarrow{OP}$, and let $S = \rho_m \circ \rho_n$. Then $S(P) = \rho_m(\rho_n(P)) = \rho_m(P) = Q$, as claimed.

Thanks to Corollary 5, we also get $S(P') = S\big(R(P)\big) = R\big(S(P)\big) = R(Q) = Q'$.

From the three equations $S(O) = O$, $S(P) = Q$, and $S(P') = Q'$, plus the fact that S is an isometry and hence preserves colinearity, the relation $\cdot * \cdot * \cdot$, and angles, the conclusion readily follows. □

Translations and glide reflections

Definition 7. An isometry which has an invariant line but no fixed point, is called a *translation* if it maps each half plane determined by that line into itself. It is called a *glide reflection* if each half plane is mapped into the opposite half plane.

In more detail, we are looking at an isometry T and a line k so that $T(P) \in k$ for every $P \in k$. Further, if the two half planes determined by k are H_1 and H_2, then T is a translation if $T(P) \in H_1$ for each $P \in H_1$, and $T(P) \in H_2$ for each $P \in H_2$. In contrast, T is a glide reflection if $T(P) \in H_2$ for each $P \in H_1$, and $T(P) \in H_1$ for each $P \in H_2$.

It is not too hard to show that any isometry with an invariant line byt no fixed point must be either a translation or a glide reflection.

Here is our version of the Translation Theorem:

Theorem 8. *If T is a translation with invariant line k, and $\ell \perp k$, there exists a line $m \perp k$ so that so that $T = \rho_m \circ \rho_\ell$. Also, for any line $m \perp k$ one can choose ℓ so that this formula holds.*

Conversely, $\rho_m \circ \rho_\ell$ is a translation with invariant line k for any pair of distinct lines $\ell \perp k$ and $m \perp k$.

Proof. Let T, k, and ℓ be as stated. Let P be the common point of k and ℓ, put $P' = T(P)$, and let m be the perpendicular bisector of $\overline{PP'}$. Then $\rho_m\big(T(P)\big) = \rho_m(P') = P$. So $\rho_m \circ T$ is an isometry with a fixed point at P. It also has k as an invariant line. Further, any $Q \in \ell$ is a fixed point of $\rho_m \circ T$: For if $R = \rho_m\big(T(Q)\big)$ then $\overleftrightarrow{RP} \perp k$ since $\rho \circ T$ preserves angles and has the fixed point P (and leaves ℓ invariant). Thus $R \in \ell$. But also Q, $T(Q)$, and R all lie on the same side of k: The first two because T is a translation, and the latter two because ρ_m maps each side of ℓ into itself. We conclude that Lemma 1 that either $\rho_m \circ T = \iota$ or $\rho_m \circ T = \rho_\ell$. So either $T = \rho_m$ or $T = \rho_m \circ \rho_\ell$. The former possibility contradicts the assumption that T has no fixed points, so the latter option is the only choice.

To see that we could have chosen the line m freely, rather than m, apply the previous result to T^{-1}.

Next, we handle the final part. It is clear that ρ_ℓ and ρ_m individually have invariant line k and map each side of k into itself. If $\rho_m \circ \rho_l$ has a fixed point P, then $\rho_\ell(P) = \rho_m(P)$. Name this common value P', and let M be the midpoint of $\overline{PP'}$. Thus $\rho_\ell(P) = P'$ and $\rho_\ell(P') = P$, and it follows that $\rho_\ell(M) = M$. Therefore, $M \in \ell$. But $M \in m$ for the same reason. This is impossible, for $\ell \parallel m$ since the two lines have a common perpendicular. □

Theorem 9. *If S is a glide reflection with invariant line k, then $S = \rho_k \circ T$ for a translation T. Conversely, this formula defines a glide reflection.*

Proof. The only non-trivial part is to show that if S is a glide reflection with invariant line k and we define $T = \rho_k \circ S$, then T is a translation. And the only non-trivial part of *that* is to show it has no fixed points. If P is such a fixed point, then $S(P) = \rho_k(P)$. Write $P' = \rho_k(P) = S(P)$, and write $P'' = S(P')$. Note that $P' \notin k$, for otherwise, $S(P) = \rho_k(P) = P$, but S has no fixed points. Then since $\overleftrightarrow{PP'} \perp k$ and S preserves angles, we have $\overleftrightarrow{P'P''} \perp k$ also. Since these two lines have the point P' in common and both are perpendicular to k, they must be the same line. In particular, the intersection point of k and $\overleftrightarrow{PP'}$ will be a fixed point of S. But that does not exist by assumption, and this completes the proof. □

Final remarks: The following remarks could easily be made into formal statements, but this is perhaps better left to the reader.

First, notice that when $\ell \perp k$ then $\rho_\ell \circ \rho_k = \rho_k \circ \rho_\ell$. So combining Theorems 8 and 9, we conclude that a glide reflection can be written

$$S = \rho_k \circ \rho_m \circ \rho_\ell = \rho_m \circ \rho_k \circ \rho_\ell = \rho_m \circ \rho_\ell \circ \rho_k$$

where $\ell \perp k$ and $m \perp k$. Here k is the (unique) invariant line of the glide reflection, and either ℓ or m can be freely chosen among perpendiculars of k.

Further, we can mimic the proof of Lemma 4 and Corollary 5 to find any two translations, or glide reflections, or one of each, with invariant line commute.

Then we could mimic the proof of Theorem 6 to show that any translation or glide reflection, when restricted to the invariant line k, simply moves points along the line by some fixed distance. But this is much more easily proved by employing the ruler axiom and noting that any $f: \mathbb{R} \to \mathbb{R}$ satisfying $|f(x) - f(y)| = |x - y|$ and having no fixed points has the form $f(x) = x + a$ for some constant $a \neq 0$.

It is quite obvious that a glide reflection has only one invariant line. For a translation in Euclidean geometry however, any line parallel to a given invariant line will itself be invariant. In hyperbolic geometry, on the other hand, a translation has only one invariant line. (This could be a good exercise.)