

## TRAINING-MA2106 HØST 2024

### 1. HOLOMORPHIC AND HARMONIC FUNCTIONS, CAUCHY-RIEMANN EQUATIONS, ELEMENTARY FUNCTIONS

**Exercise 1.1.** Use either the definition of complex-differentiability or the Cauchy Riemann Equations to show that:

- (a) The function  $f(z) = |z|^2$  is differentiable only at  $z = 0$ , with  $f'(0) = 0$ .
- (b) The function  $g(z) = |z|$  is not differentiable at any  $z \in \mathbb{C}$ .

**Exercise 1.2.** Let  $\Omega \subset \mathbb{C}$  open and  $f : \Omega \rightarrow \mathbb{C}$  differentiable at  $z_0 \in \Omega$  with  $f'(z_0) \neq 0$ . Prove that there exists  $r > 0$  so that  $f(z) \neq f(z_0)$  for all  $z \in D(z_0, r)$ .

**Exercise 1.3.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  the function defined by

$$f(x + iy) = x^2 + 2y + i(x^2 + y^2), \quad \text{for all } x + iy \in \mathbb{C}.$$

Determine at which  $z \in \mathbb{C}$  the function  $f$  is differentiable.

**Exercise 1.4.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be holomorphic so that  $f(0) = i$  and the real part  $u = \operatorname{Re}(f)$  is

$$u(x + iy) = 2x^3y - 2xy^3 + x^2 - y^2 \quad \text{for all } x + iy \in \mathbb{C}.$$

Find  $v = \operatorname{Im}(f)$ .

**Exercise 1.5 (Exam December 2023).** Prove that the function  $f : \mathbb{C} \rightarrow \mathbb{C}$  given by  $f(z) = (\bar{z})^2$  for all  $z \in \mathbb{C}$  is complex-differentiable only at  $z = 0$  and show that  $f'(0) = 0$ .

**Exercise 1.6.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  differentiable at  $z_0$  with  $f'(z_0) \neq 0$ . Prove that  $\bar{f} := \operatorname{Re}(f) - i \operatorname{Im}(f)$  is not differentiable at  $z_0$ .

**Exercise 1.7 (Exam December 2022).** The tasks are:

- (a) Show that the function

$$u(x, y) = x^3 - 3xy^2 + e^{-x} \cos y, \quad (x, y) \in \mathbb{R}^2,$$

is harmonic.

- (b) Find the harmonic conjugate  $v$  of  $u$  that satisfies  $v(0, 0) = 0$ .

**Exercise 1.8.** Let  $\Omega \subset \mathbb{C}$  be open and connected, and  $f : \Omega \rightarrow \mathbb{C}$  holomorphic. Show that  $f$  is constant in  $\Omega$  in each of the following situations:

- (a)  $f(\Omega) \subset \mathbb{R}$  ( $f$  takes only real values) or  $f(\Omega) \subset i\mathbb{R}$  ( $f$  takes only pure imaginary values).
- (b)  $\operatorname{Re}(f) : \Omega \rightarrow \mathbb{R}$  or  $\operatorname{Im}(f) : \Omega \rightarrow \mathbb{R}$  is constant in  $\Omega$ .
- (c)  $\bar{f} := \operatorname{Re}(f) - i \operatorname{Im}(f)$  is holomorphic in  $\Omega$ .
- (d) The modulus of  $|f|$  is constant in  $\Omega$ .

(e) The principal argument of  $f$ ,  $\Omega \ni z \mapsto \text{Arg}(f)(z) := \text{Arg}(f(z))$  is constant in  $\Omega$ . Here, we additionally assume that  $f(z) \neq 0$  for all  $z \in \Omega$ .

**Exercise 1.9.** Show that if  $f : \Omega \rightarrow \mathbb{R}$  is differentiable at  $z_0 = x_0 + iy_0 \in \Omega$ , then

$$|f'(z_0)| = \|\nabla \text{Re}(f)(x_0, y_0)\| = \|\nabla \text{Im}(f)(x_0, y_0)\|;$$

where  $\|\cdot\|$  is the Euclidean norm in  $\mathbb{R}^2$ ; that is,  $\|(a, b)\| = \sqrt{a^2 + b^2}$ .

**Exercise 1.10 (Øving November 2022).** Show that the function

$$u(x, y) = xe^x \cos y - ye^x \sin y, \quad (x, y) \in \mathbb{R}^2,$$

is harmonic. Then find the harmonic conjugate  $v$  of  $u$  that satisfies  $v(0, 0) = 0$ .

**Exercise 1.11.** Use the Cauchy-Riemann Equations to show that the function

$$f(z) = \bar{z}, \quad z \in \mathbb{C}$$

is not complex-differentiable at any  $z \in \mathbb{C}$ .

**Exercise 1.12.** Let  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the function  $u(x, y) = xy$ . Prove that  $u$  is harmonic in  $\mathbb{R}^2$  and find a harmonic conjugate  $v$  of  $u$  with  $v(0, 0) = 0$ .

**Exercise 1.13.** Let  $\Omega \subset \mathbb{C}$  be open and connected and  $u : \Omega \rightarrow \mathbb{R}$  be harmonic. Show that if  $v$  and  $\tilde{v}$  are two harmonic conjugates of  $u$  in  $\Omega$ , then  $v - \tilde{v}$  is a constant function.

**Exercise 1.14.** Let  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a harmonic function. Show that the function  $f := \frac{\partial u}{\partial y} + i \frac{\partial u}{\partial x}$  is holomorphic in  $\mathbb{C}$ .

**Exercise 1.15.** For the function  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by

$$u(x, y) = y^3 - 3x^2y, \quad (x, y) \in \mathbb{R}^2,$$

prove that  $u$  is harmonic in  $\mathbb{R}^2$  and find the harmonic conjugate  $v$  of  $u$  that satisfies  $v(0, 0) = 1$ .

**Exercise 1.16.** Define  $u : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{R}$  by  $u(z) = \log |z|$ . Prove that  $u$  is (real) harmonic in  $\mathbb{C} \setminus \{0\}$ , and that  $u$  has no harmonic conjugate in  $\mathbb{C} \setminus \{0\}$ .

*Suggestion: Write  $u(x, y)$  in a simple form. Suppose that  $v : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{R}$  is a harmonic conjugate of  $u$ . The Cauchy-Riemann equations for  $u + iv$  will lead you to a contradiction.*

**Exercise 1.17.** For the function  $f : \mathbb{C} \rightarrow \mathbb{C}$  given by

$$f(z) = e^{z^3}, \quad z \in \mathbb{C},$$

find  $\text{Re}(f)$  and  $\text{Im}(f)$ , that is, the real and imaginary parts of  $f$ .

**Exercise 1.18 (Exam December 2023).** For the function  $f : \mathbb{C} \rightarrow \mathbb{C}$  given by

$$f(z) = e^{e^z}, \quad z \in \mathbb{C},$$

find  $\text{Re}(f)$  and  $\text{Im}(f)$ , that is, the real and imaginary parts of  $f$ .

**Exercise 1.19.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be defined by

$$f(z) = \begin{cases} e^{-1/z^4} & \text{if } z \in \mathbb{C} \setminus \{0\} \\ 0 & \text{if } z = 0. \end{cases}$$

Prove that  $f \in \mathcal{H}(\mathbb{C} \setminus \{0\})$ , that the partial derivatives  $\frac{\partial \operatorname{Re}(f)}{\partial x}$ ,  $\frac{\partial \operatorname{Re}(f)}{\partial y}$ ,  $\frac{\partial \operatorname{Im}(f)}{\partial x}$ ,  $\frac{\partial \operatorname{Im}(f)}{\partial y}$  exist and satisfy the Cauchy-Riemann equations at  $z_0 = 0$ , and that  $\lim_{z \rightarrow 0} f(z)$  does not even exist. In particular,  $f$  is not continuous at  $z_0 = 0$ .

**Exercise 1.20 (Exam December 2022).** Let  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a harmonic function in  $\mathbb{R}^2$ , and let  $v : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a harmonic conjugate. Prove that the function

$$w = e^{u^2 - (v+1)^2} \cos(2u(v+1))$$

is harmonic in  $\mathbb{R}^2$  as well.

**Exercise 1.21.** Show that  $\operatorname{Log}(1+i)^2 = 2\operatorname{Log}(1+i)$  and  $\operatorname{Log}(-1+i)^2 \neq 2\operatorname{Log}(-1+i)$ .

**Exercise 1.22.** Given  $w \in \mathbb{C}$ , show that the set of all the solutions  $z \in \mathbb{C}$  of the equation  $\sin z = w$  is  $\{-i\xi : \xi \in \langle \log(iw + \varphi) \rangle, \varphi \in \langle \sqrt{1-w^2} \rangle\}$ . In particular, write down all the solutions  $z \in \mathbb{C}$  of the equation  $\sin z = 2$ .

**Exercise 1.23.** Compute the following sets and/or numbers.

- (a) The real and imaginary parts of  $e^{3-i}$ .
- (b) The real and imaginary parts of  $\cos(2+3i)$ .
- (c)  $\langle \log(-1+\sqrt{3}i) \rangle$  and  $\operatorname{Log}(-1+\sqrt{3}i)$ .
- (d)  $\langle (-1)^i \rangle$  and  $(-1)^i$ .
- (e)  $\langle (1+i)^{(1+i)} \rangle$  and  $(1+i)^{1+i}$ .

2. SEQUENCES AND SERIES OF FUNCTIONS. POWER SERIES. ANALYTIC FUNCTION

**Exercise 2.1.** Consider the sequence of functions  $\{f_n : \mathbb{C} \rightarrow \mathbb{C}\}_n$  given by

$$f_n(z) = \frac{n + e^z}{1 + n|z|^2}, \quad n \in \mathbb{N}, z \in \mathbb{C}.$$

Find  $f : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$  so that  $\{f_n\}_n$  converges pointwise to  $f$  on  $\mathbb{C} \setminus \{0\}$ . Then show that this convergence is uniform on each set  $A_R := \{z \in \mathbb{C} : 1/R \leq |z| \leq R\}$ , with  $R > 0$ .

*Suggestion:* We remind that  $|e^z| = e^{\operatorname{Re}(z)}$ , which helps when estimating.

**Exercise 2.2.** Consider the series of functions

$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{n^2}, \quad z \in \mathbb{C}.$$

Prove that:

- (a) The series converges uniformly in  $z \in \mathbb{R}$ .
- (b) For each  $z \in \mathbb{C} \setminus \mathbb{R}$ , the numerical series  $\sum_{n=1}^{\infty} \frac{\sin(nz)}{n^2}$  diverges.

*Suggestion:* In part (b), it might be helpful to study the modulus of the general term distinguishing the cases  $\operatorname{Im}(z) > 0$  and  $\operatorname{Im}(z) < 0$ .

**Exercise 2.3.** If  $\Omega := \{z \in \mathbb{C} : \operatorname{Re}(z) > 1\}$ , prove the following about the series of functions  $\sum_{n=1}^{\infty} \frac{1}{n^z}$ ,  $z \in \Omega$ :

- (a)  $\sum_{n=1}^{\infty} \frac{1}{n^z}$  converges absolutely for each  $z \in \Omega$ .
- (b)  $\sum_{n=1}^{\infty} \frac{1}{n^z}$  converges absolutely-uniformly on each set  $\{z \in \mathbb{C} : \operatorname{Re}(z) \geq 1 + \varepsilon\}$ , with  $\varepsilon > 0$ .
- (c)  $\sum_{n=1}^{\infty} \frac{1}{n^z}$  does not converge uniformly on  $\Omega$ .

*Clarification:* Here  $n^z$  is the principal  $z$ -power of  $n$ , that is,  $n^z = e^{z \operatorname{Log} n} = e^{z \log n}$ . For parts (b) and (c), it's perhaps easier to prove/disprove the corresponding Cauchy property.

**Exercise 2.4.** Determine the disk and the radius of convergence of the following power series.

- a)  $\sum_{n=1}^{\infty} \frac{1}{n^2} z^n$
- b)  $\sum_{n=1}^{\infty} \frac{(3 + 4i)^n}{(1 + \frac{i}{n})^{n^3}} (z - i)^n$
- c)  $\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n (z - 1)^n$
- d)  $\sum_{n=1}^{\infty} n! z^n$
- e)  $\sum_{n=1}^{\infty} (2 + (-1)^n)^n z^n$
- f)  $\sum_{n=1}^{\infty} (\log n)^2 z^n$
- g)  $\sum_{n=1}^{\infty} \frac{n^n}{1 + 2^n n^n} z^n$
- h)  $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}} z^n$
- i)  $\sum_{n=1}^{\infty} z^{n^2}$
- j)  $\sum_{n=1}^{\infty} 2^n z^{n!}$
- k)  $\sum_{n=1}^{\infty} \frac{1}{n^2} z^{n!}$
- l)  $\sum_{n=1}^{\infty} 3^{n^2} z^{1+2+\dots+n}$

**Exercise 2.5.** For the series a)–l) in Exercise 2.4, study the convergence in the boundary of their disk of convergence.

**Exercise 2.6.** For every  $m \in \mathbb{N}$ , find a power series with disk of convergence  $D(0, 1)$  and so that it diverges precisely at  $m$  points of the boundary  $\partial D(0, 1)$  of  $D(0, 1)$ .

**Exercise 2.7.** Let  $f \in \mathcal{A}(D(0, 1))$  such that  $f(\frac{1}{n^2}) = \frac{n^4}{(n^2-1)^2}$  for all  $n \in \mathbb{N}$ ,  $n \geq 2$ . Find the explicit formula for  $f(z)$  for all  $z \in D(0, 1)$ .

**Exercise 2.8.** Let  $f \in \mathcal{A}(D(0, 1))$  such that  $f(\frac{1}{n}) = \frac{n^2}{n^2+1}$  for all  $n \in \mathbb{N}$ ,  $n \geq 2$ . Find the explicit formula for  $f(z)$  for all  $z \in D(0, 1)$ , and calculate  $f^{(n)}(0)$  for all  $n \in \mathbb{N}$ .

### 3. CAUCHY INTEGRAL THEOREMS AND FORMULAE. WINDING NUMBERS

**Exercise 3.1.** Let  $\Omega$  be open and convex, and  $f : \Omega \rightarrow \mathbb{C}$  holomorphic (with  $f'$  continuous) in  $\Omega$  with  $\operatorname{Re}(f'(z)) > 0$  for all  $z \in \Omega$ . Show that  $f$  is injective in  $\Omega$ .

**Exercise 3.2.** Let  $\Omega \subset \mathbb{C}$  open,  $f : \Omega \rightarrow \mathbb{C}$  holomorphic and  $\gamma : [a, b] \rightarrow \Omega$  a closed piecewise  $C^1$ -path. Prove that if  $n \in \mathbb{N} \cup \{0\}$  and  $z_0 \notin \gamma^*$ , then

$$\int_{\gamma} \frac{f'(z)}{(z - z_0)^n} dz = n \int_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

*Suggestion: Use appropriately integration by parts.*

**Exercise 3.3.** Let  $z_1, z_2 \in \mathbb{C}$  and  $r > 0$  so that  $|z_1| < r < |z_2|$ . Show that

$$\int_{\partial D(0,r)} \frac{1}{(z - z_1)(z - z_2)} dz = \frac{2\pi i}{z_1 - z_2}.$$

**Exercise 3.4.** Compute the following path-integrals in the indicated sets/paths.

(a)  $\int_{\gamma} \frac{z^2+1}{z(z^2+4)} dz$ , where  $\gamma = \partial D(0, r)$  for  $r \neq 2$ .

(b)  $\int_{\gamma} \frac{e^{1/z}}{z^2+z} dz$ , where  $\gamma = \partial D(-1, 1/2)$ .

(c)  $\int_{\gamma} \frac{z^2+7z+6}{(z+2)(z+4)} dz$ ; where  $\gamma = \partial D(0, 3)$ .

(d)  $\int_{\gamma} \frac{\sin(e^z)}{z} dz$ , where  $\gamma = \partial D(0, 1)$ .

(e)  $\int_{\gamma} \frac{e^z}{z^2(z-1)} dz$ , where  $\gamma = \partial D(0, 2)$ .

(f)  $\int_{\gamma} z^2 \sin\left(\frac{1}{z}\right) dz$ , where  $\gamma = \partial D(0, 1)$ .

*Suggestion: In (f), recall the Taylor expansion of the complex sin function.*

**Exercise 3.5.** Prove that:

(a)  $\int_{\gamma} \frac{3z-1}{(z+1)(z-3)} dz = 6\pi i$ , where  $\gamma = \partial D(0, 4)$ .

(b)  $\int_{\gamma} \frac{2z}{z^2+1} dz = 4\pi i$ , where  $\gamma = \partial D(0, 2)$ .

(c)  $\int_{\gamma} \frac{z^3}{z^4-1} dz = 2\pi i$ , where  $\gamma = \partial D(0, 2)$ .

(d)  $\int_{\gamma} \frac{e^z}{(z-2)^2} dz = 2\pi i e^2$ , where  $\gamma = \partial D(2, 1)$ .

**Exercise 3.6.** Let  $r > 0$  and let  $P$  be a polynomial whose roots are all contained in the open disk  $D(0, r)$ . Prove that

$$\int_{\partial D(0,r)} \frac{P'(z)}{P(z)} dz = 2\pi i \operatorname{deg}(P).$$

*Suggestion: Use the Fundamental Theorem of Algebra to factorize  $P$ , and then observe the partial fraction decomposition of  $P'/P$ .*

**Exercise 3.7.** Let  $\Omega \subset \mathbb{C}$  be open so that  $\overline{D}(z_0, r) \subset \Omega$ , and let  $f : \Omega \rightarrow \mathbb{C}$  be holomorphic. Prove that

$$|f(z_0)|^2 \leq \frac{1}{\pi r^2} \int_0^{2\pi} \int_0^r |f(z_0 + se^{it})|^2 s ds dt.$$

*Suggestion: Use the Mean Value property for  $f^2$  on each circle  $\partial D(z_0, s)$  with  $0 < s \leq r$ .*

**Exercise 3.8.** Use appropriately the Cauchy Integral Formula to calculate the integral

$$\int_{\partial D(0,1)} \frac{1}{(z - 3/2)(z - 2/3)} dz.$$

Then calculate the real integral

$$\int_0^{2\pi} \frac{dt}{13 - 12 \cos t}.$$

**Exercise 3.9 (Exam December 2022).** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be holomorphic in  $\mathbb{C}$  so that

$$|f(re^{i\theta})| \leq \sqrt{\frac{r}{|\theta|}}, \quad \text{whenever } 0 < |\theta| \leq \pi, r \geq 100.$$

Prove that  $f$  is constant in  $\mathbb{C}$ .

*Suggestion: Use the Cauchy Integral formula (not the Cauchy Estimates) for  $f'$*

4. CAUCHY ESTIMATES FOR THE DERIVATIVES. LIOUVILLE'S THEOREM. IDENTITY PRINCIPLES. MAXIMUM MODULUS PRINCIPLES

**Exercise 4.1.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  holomorphic such that there constants  $C, a > 0$  with  $|f(z)| \leq Ce^{a|z|}$  for all  $z \in \mathbb{C}$ . Prove that  $|f'(z)| \leq Cae^{a|z|+1}$  for all  $z \in \mathbb{C}$ .

*Suggestion: Use the Cauchy Estimates*

**Exercise 4.2.** Let  $f : D(0, 1) \rightarrow \mathbb{C}$  be holomorphic. Prove that

$$2|f'(0)| \leq \sup\{|f(w) - f(-w)| : w \in \mathbb{D}\}.$$

*Suggestion: Consider  $g(w) := f(-w)$  and apply the Cauchy Estimates for  $h := f - g$ .*

**Exercise 4.3 (Exam August 2023, slightly modified).** The tasks are the following.

(a) Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a holomorphic function so that there is  $m \in \mathbb{N}$  with the property that

$$|f(z)| \leq (1 + |z|)^m, \quad \text{for all } z \in \mathbb{C}.$$

*Prove that  $f$  is a polynomial of degree at least  $m$ .*

*Suggestion: Use the Cauchy Estimates for the derivatives of  $f$*

(b) Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a holomorphic function in  $\mathbb{C}$  so that

$$\operatorname{Re}(f(z)) \leq \log(1 + |z|) \quad \text{for all } z \in \mathbb{C}.$$

*Prove that  $f$  is constant in  $\mathbb{C}$ .*

*Suggestion: Use part (a)*

**Exercise 4.4.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be holomorphic, and suppose there are constants  $r, C > 0, n \in \mathbb{N}$  such that  $|f(z)| \leq C|z|^n$  for all  $|z| \geq r$ . Prove that  $f$  is a polynomial with  $\deg(f) \leq n$ .

*Suggestion: Use the Cauchy Estimates for the derivatives of  $f$*

**Exercise 4.5.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be holomorphic with  $|f'(z)| \leq |z|$  for all  $z \in \mathbb{C}$ . Show that there exist  $w_1, w_2 \in \mathbb{C}$  with  $|w_2| \leq 1/2$  so that  $f(z) = w_1 + w_2 z^2$  for all  $z \in \mathbb{C}$ .

*Suggestion: First use the Cauchy Estimates for the derivatives of  $f''$  to show that  $f'$  is a polynomial of degree 1.*

**Exercise 4.6.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be holomorphic, and suppose there are constants  $C > 0, \alpha > 0$  with  $\alpha \notin \mathbb{N}$  such that  $|f(z)| \leq C|z|^\alpha$  for all  $z \in \mathbb{C}$ . Prove that  $f$  is identically zero in  $\mathbb{C}$ .

*Suggestion: First use the Cauchy Estimates for the derivatives  $f^{(k)}(0)$ . Then take into account the First Identity Principle for Holomorphic Functions.*

**Exercise 4.7.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be holomorphic. Prove that for every  $z \in \mathbb{C}$  there exists  $M > 0$  such that  $|f^{(n)}(z)| \leq M \cdot n!$  for every  $n \in \mathbb{N}$ .

*Suggestion: Write  $f$  as a power series in  $\mathbb{C}$ . What is the radius of convergence of that series?*

**Exercise 4.8.** Let  $\Omega$  be open,  $z_0 \in \Omega$ , and  $f : \Omega \rightarrow \mathbb{C}$  holomorphic in  $\Omega$ . Show that the estimates

$$|f^{(n)}(z_0)| \geq n! n^n$$

*only can hold for finitely many  $n \in \mathbb{N}$ .*

*Suggestion: Write  $f$  as a power series in a disk centered at  $z_0$ . Then argue by contradiction.*

**Exercise 4.9.** Let  $\Omega$  be open and connected, and  $f : \Omega \rightarrow \mathbb{C}$  be holomorphic so that there exists  $z_0 \in \Omega$  and  $C > 0$  with  $|f^{(n)}(z_0)| \leq C$  for every  $n \in \mathbb{N}$ . Prove that there is a function  $g : \mathbb{C} \rightarrow \mathbb{C}$  holomorphic with  $g = f$  on  $\Omega$ .

*Suggestion:* What is the radius of convergence of the power series  $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$ ? Then, take into account the Second Identity Principle for Holomorphic Functions.

**Exercise 4.10.** Let  $\Omega$  be open and connected, and  $f, g : \Omega \rightarrow \mathbb{C}$  be two holomorphic functions with  $f(z)g(z) = 0$  for all  $z \in \Omega$ . Show that either  $f = 0$  in  $\Omega$  or  $g = 0$  in  $\Omega$ .

*Suggestion:* Use appropriately Second Identity Principle for Holomorphic Functions.

**Exercise 4.11.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be holomorphic with  $\lim_{|z| \rightarrow \infty} \frac{f(z)}{z} = 0$ . Prove that  $f$  is constant in  $\mathbb{C}$ .

*Suggestion:* First Use the Cauchy Estimates to show that  $f$  is a polynomial of degree 1.

**Exercise 4.12.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be holomorphic so that there exist  $M > 0$ ,  $\alpha \in (0, 1)$  with  $|f(z)| \leq M(1 + |z|^\alpha)$  for all  $z \in \mathbb{C}$ . Prove that  $f$  is constant in  $\mathbb{C}$ .

*Suggestion:* Use the Cauchy estimates for  $|f^{(n)}(0)|$  and then take into account the First Identity Principle for holomorphic functions.

**Exercise 4.13.** Show that there exist **no** function  $f : D(0, 2) \rightarrow \mathbb{C}$  holomorphic in  $D(0, 2)$  with  $f(1/n) = -1/n^2$  and  $f\left(\frac{n+1}{n}\right) = 1/n$  for all  $n \geq 2$ .

*Suggestion:* Use appropriately the Second Identity Principle for holomorphic functions.

**Exercise 4.14.** Show that there exist **no** function  $f : D(0, 2) \rightarrow \mathbb{C}$  holomorphic in  $D(0, 2)$  with

$$f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n^2}, \quad \text{for all } n \in \mathbb{N}.$$

*Suggestion:* Use appropriately the Second Identity Principle for holomorphic functions.

**Exercise 4.15.** Let  $\mathbb{D} = D(0, 1)$  be the unit disk, and let  $f : \mathbb{D} \rightarrow \mathbb{C}$  be holomorphic.

(a) Prove that the function  $g : \mathbb{D} \rightarrow \mathbb{C}$  given by

$$g(z) = \overline{f(\bar{z})}, \quad \text{for all } z \in \mathbb{D},$$

is holomorphic in  $\mathbb{D}$  as well.

(b) Assume further that  $f\left(\frac{1}{n+1}\right) \in \mathbb{R}$  for all  $n \in \mathbb{N}$ . Prove that then

$$f(z) = g(z), \quad \text{for all } z \in \mathbb{D},$$

where  $g$  is the function from (a).

*Suggestion:* In part (b), use appropriately the Identity Principle II for Holomorphic Functions.

**Exercise 4.16 (Øving November 2022; slightly modified).** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be holomorphic in  $\mathbb{C}$  so that there exists  $M > 0$  with the property that

$$\operatorname{Re}(f(z)) \leq M, \quad \text{for all } z \in \mathbb{C}.$$

Prove that  $f$  is constant in  $\mathbb{C}$ .

*Suggestion:* Liouville's Theorem

**Exercise 4.17 (Exam December 2023).** Let  $f : \Omega \rightarrow \mathbb{C}$  be a holomorphic function in the domain  

$$\Omega := \mathbb{C} \setminus \{iy : y \leq 0\}.$$

Assume that  $f(x) = x \log x$  for all  $0 < x < 1$ . Find the explicit formula of  $f(z)$  for all  $z \in \Omega$ , and, in particular evaluate  $f(-1 - i)$ .

*Suggestion: The Identity Principle II for holomorphic functions.*

**Exercise 4.18.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be holomorphic. Prove that  $f$  is constant in any of the following cases.

- (i)  $|f(z)| \geq 1$  for all  $z \in \mathbb{C}$ .
- (ii) Either  $\operatorname{Re}(f)(z) \geq 0$  for all  $z \in \mathbb{C}$  or  $\operatorname{Im}(f)(z) \geq 0$  for all  $z \in \mathbb{C}$ .
- (iii) Either  $\operatorname{Re}(f)(z) \neq 0$  for all  $z \in \mathbb{C}$  or  $\operatorname{Im}(f)(z) \neq 0$  for all  $z \in \mathbb{C}$ .

*Suggestion: Use appropriately Liouville's Theorem.*

**Exercise 4.19.** For any nonempty set  $A \subset \mathbb{C}$ , denote  $M(A) := \max\{|e^z| : z \in A\}$ . Find  $M(A)$  in the following cases:

- (i)  $A = \{z \in \mathbb{C} : |z - 1 - i| \leq 1\}$ .
- (ii)  $A = \{z \in \mathbb{C} : |\operatorname{Re}(z) - 2| \leq 1 \text{ and } |\operatorname{Im}(z) - 3| \leq 1\}$ .

*Suggestion: Use the Maximum Modulus Principle II*

**Exercise 4.20.** For any nonempty set  $A \subset \mathbb{C}$ , denote  $M(A) := \max\{|\cos z| : z \in A\}$ . Find  $M(A)$  in the following cases:

- (i)  $A = \{z \in \mathbb{C} : \operatorname{Re}(z), \operatorname{Im}(z) \in [0, 2\pi]\}$ .
- (ii)  $A = \{z \in \mathbb{C} : |z| \leq 1\}$ .

*Suggestion: Use the Maximum Modulus Principle II*

**Exercise 4.21.** Denote  $\mathbb{D} = D(0, 1)$  and  $\overline{\mathbb{D}} = \overline{D}(0, 1)$ . Let  $\Omega \subset \mathbb{C}$  be open and connected so that  $\overline{\mathbb{D}} \subset \Omega$  and let  $f : \Omega \rightarrow \mathbb{C}$  be holomorphic. Prove the following statements.

- (a) If  $f$  is non-constant in  $\Omega$  and  $|f|$  is constant in  $\partial\mathbb{D}$ , then there exists  $z_0 \in \mathbb{D}$  with  $f(z_0) = 0$ .  
 Suggestion: Assuming that  $f(z) \neq 0$  for all  $z \in \mathbb{D}$ , use the Maximum Modulus Principle II for both  $f$  and  $1/f$  to deduce that  $|f|$  is constant in  $\mathbb{D}$ . Then deduce that  $f$  is constant in  $\mathbb{D}$ , and arrive at a contradiction.
- (b) If  $f$  is pure imaginary in  $\partial\mathbb{D}$  (that is,  $f(\partial\mathbb{D}) \subset i\mathbb{R}$ ), then  $f$  is constant in  $\Omega$ .  
 Suggestion: Use (a) for an appropriate variation of  $f$ .
- (c) If  $f$  is real in  $\partial\mathbb{D}$  (that is,  $f(\partial\mathbb{D}) \subset \mathbb{R}$ ), then  $f$  is constant in  $\Omega$ .  
 Suggestion: Use (b) for an appropriate variation of  $f$ .

5. SINGULARITIES. RESIDUES. THE CAUCHY RESIDUES THEOREM. EVALUATION OF INTEGRALS

**Exercise 5.1 (Exam August 2024).** Let  $f : \mathbb{C} \setminus \{0\}$  be holomorphic in  $\mathbb{C} \setminus \{0\}$  so that there is a constant  $M > 0$  with

$$|f(z)| \leq M, \quad \text{whenever } 0 < |z| \leq 1.$$

Assume also that there is a sequence  $\{a_n\}_{n \in \mathbb{N}} \subset \mathbb{C} \setminus \{0\}$  with

$$\lim_{n \rightarrow \infty} a_n = 0, \quad \text{and} \quad \lim_{n \rightarrow \infty} f(a_n) = 0.$$

Show that

(a) There exists  $w_0 \in \mathbb{C}$  such that

$$\lim_{z \rightarrow 0} \frac{f(z)}{z} = w_0.$$

Suggestion: What kind of singularity does  $f$  have at 0? Then recall Riemann's Theorem.

(b) One has  $|w_0| \leq M$ .

Suggestion: You can use the Cauchy Estimates for  $|f'(0)|$ .

**Exercise 5.2 (Exam December 2023).** Let  $f : \mathbb{C} \setminus \{0\}$  be holomorphic in  $\mathbb{C} \setminus \{0\}$ , and let  $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$  satisfying

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 0, \quad \lim_{n \rightarrow \infty} f(a_n) = 0, \quad \lim_{n \rightarrow \infty} f(b_n) = 1.$$

Prove that for every  $w \in \mathbb{C}$  there exists  $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$  with  $\lim_{n \rightarrow \infty} f(z_n) = w$ .

Suggestion: Recall Casorati-Weierstrass theorem

**Exercise 5.3 (Exam August 2024).** Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{2 - \cos \theta}.$$

**Exercise 5.4 (Exam December 2022).** Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos \theta}.$$

**Exercise 5.5.** Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{9 + 16 \sin^2(\theta)}.$$

**Exercise 5.6 (Øving November 2022).** The function

$$f(z) = \frac{e^{1/z}}{1-z}$$

has singularities at  $z = 0$  and  $z = 1$ . Describe the type of singularities of  $f$  at those points.

Then, evaluate the complex path-integral

$$\int_{\partial D(0,2)} f(z) dz;$$

where  $\partial D(0,2)$  is traveled once and counterclockwise.

**Exercise 5.7 (Exam December 2022).** The function

$$f(z) = \frac{\sin\left(\frac{\pi}{z}\right)}{1-z}$$

has singularities at  $z = 0$  and  $z = 1$ . Describe the type of singularities of  $f$  at those points.

Then, evaluate the complex path-integral

$$\int_{\partial D(0,2)} f(z) \, dz;$$

where  $\partial D(0,2)$  is traveled once and counterclockwise.

**Exercise 5.8.** For every  $n \in \mathbb{N} \cup \{0\}$ , evaluate the integral

$$\int_{\partial D(0,1)} z^n e^{1/z} \, dz;$$

where  $\partial D(0,1)$  is traveled once and counterclockwise.

**Exercise 5.9.** Prove that

$$\int_{\partial D(0,1)} e^{z+\frac{1}{z}} \, dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n! \cdot (n+1)!}.$$

**Exercise 5.10.** For the ellipse  $\gamma(t) = \{a \cos t + ib \sin t : t \in [0, 2\pi]\}$ ,  $a, b > 0$ , evaluate

$$\int_{\gamma} \frac{e^{-z^2}}{z^2} \, dz.$$

**Exercise 5.11.** Using Cauchy Residues Theorem, evaluate the following path-integrals, always traveled once and with the counterclockwise orientation.

(a)  $\int_{\gamma} \frac{1}{z^4+1} \, dz$ , where  $\gamma$  is the ellipse  $x^2 - xy + y^2 + x + y = 0$ .

(b)  $\int_{\gamma} \frac{\cos(\frac{z}{2})}{z^2-4} \, dz$ , where  $\gamma$  is the ellipse  $x^2/9 + y^2/4 = 1$ .

(c)  $\int_{\gamma} \frac{z}{z^2+2z+5} \, dz$ , where  $\gamma \equiv \partial D(0,1)$ .

(d)  $\int_{\gamma} \frac{e^z}{(1-z)^3} \, dz$ , where  $\gamma \equiv \partial D(1,1/2)$ .

(e)  $\int_{\gamma} \frac{1}{z(z-1)(z-2)} \, dz$ , where  $\gamma \equiv \partial D(0,3/2)$ .

(f)  $\int_{\gamma} \frac{1+z}{1-\cos z} \, dz$ , where  $\gamma \equiv \partial D(0,7)$ .

**Exercise 5.12.** Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}.$$

**Exercise 5.13.** Find a closed formula (in terms of the parameter  $a$ ) for the following the integral

$$\int_0^{2\pi} \frac{d\theta}{a+\sin\theta},$$

when  $a > 1$ , and also when  $a < -1$ .

**Exercise 5.14.** Show that

$$\int_0^\pi \frac{d\theta}{1 + \sin^2 \theta} = \frac{\pi}{\sqrt{2}}.$$

**Exercise 5.15.** For  $a, b > 0$ , with  $a \neq b$ , evaluate the integral

$$\text{pv} \int_{-\infty}^{+\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} dx.$$

**Exercise 5.16.** For  $a > 0$ , evaluate the integral

$$\text{pv} \int_{-\infty}^{+\infty} \frac{\cos x}{(x^2 + a^2)^2} dx.$$

**Exercise 5.17.** Evaluate the integral

$$\text{pv} \int_{-\infty}^{+\infty} \frac{dx}{x^2 - 2x + 4}.$$

**Exercise 5.18.** Evaluate the integral

$$\text{pv} \int_{-\infty}^{+\infty} \frac{\sin x}{x^2 - 2x + 2} dx.$$

**Exercise 5.19.** Evaluate the integral

$$\text{pv} \int_{-\infty}^{+\infty} \frac{x^3 \sin x}{(x^2 + 1)^2} dx.$$

**Exercise 5.20.** Prove the identity

$$\text{pv} \int_{-\infty}^{+\infty} \frac{x \sin x}{x^4 + 1} dx = \pi e^{-\frac{1}{\sqrt{2}}} \sin\left(\frac{1}{\sqrt{2}}\right).$$

**Exercise 5.21.** Evaluate the integral

$$\text{pv} \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^3 - 2x^2 + x - 2} dx,$$

and then

$$\text{pv} \int_{-\infty}^{+\infty} \frac{\cos x}{x^3 - 2x^2 + x - 2} dx,$$

**Exercise 5.22.** Evaluate the integral

$$\text{pv} \int_{-\infty}^{+\infty} \frac{\sin x}{x(x^2 - 1)} dx.$$

**Exercise 5.23.** Evaluate the integral

$$\text{pv} \int_{-\infty}^{+\infty} \frac{e^{ix}}{x(x - w_0)^2} dx,$$

where  $w_0 \in \mathbb{C}$  is so that  $\text{Im}(w_0) > 0$ .

**Exercise 5.24.** Evaluate the integral

$$\int_0^{2\pi} \frac{e^{4i\theta}}{2 + \cos \theta} d\theta,$$

and then deduce the value of

$$\int_0^{2\pi} \frac{\cos(4\theta)}{2 + \cos \theta} d\theta.$$

6. FOURIER SERIES. FOURIER SERIES OF SINES AND COSINES. DIFFERENTIAL EQUATIONS

**Exercise 6.1.** For the following functions: consider their  $2\pi$ -periodic extensions to all of  $\mathbb{R}$ , and find their Fourier Coefficients, their Fourier Sums, and their Fourier Series. Then, for all  $x \in [0, 2\pi)$ , study whether or not the Fourier Series  $Sf(x)$  converges to  $f(x)$ .

(i) The function  $f : [0, 2\pi) \rightarrow \mathbb{R}$  defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \pi] \\ -1 & \text{if } x \in [\pi, 2\pi). \end{cases}$$

(ii) The function  $f(x) = |x - \pi|$  for all  $x \in [0, 2\pi]$ .

(iii) The function  $f(x) = x + \sin x + \cos x$ , for all  $x \in [0, 2\pi)$ .

(iv) The function  $f(x) = (x - \pi)^2$ , for all  $x \in [0, 2\pi]$ .

**Exercise 6.2.** Calculate the Fourier Transform of the following functions  $f : \mathbb{R} \rightarrow \mathbb{C}$  :

(i) For  $a > 0$ ,

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq a \\ 0 & \text{if } |x| > a. \end{cases}$$

(ii) For  $a > 0$ ,

$$f(x) = \begin{cases} e^{-ax} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

(iii) For  $a > 0$ ,  $f(x) = e^{-a|x|}$  for all  $x \in \mathbb{R}$ .

(iv) For  $a > 0$ ,

$$f(x) = \begin{cases} \frac{x^2}{2} e^{-ax} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

**Exercise 6.3.** For a function  $f : [0, \pi] \rightarrow \mathbb{R}$ , denote the Fourier coefficients

$$a_0 := \frac{1}{\pi} \int_0^\pi f(t) dt, \quad a_n := \frac{2}{\pi} \int_0^\pi f(t) \cos(nt) dt, \quad b_n := \frac{2}{\pi} \int_0^\pi f(t) \sin(nt) dt, \quad n \in \mathbb{Z} \setminus \{0\}.$$

The Fourier Series of Cosines and the Fourier Series of Sines of  $f$  in  $[0, \pi]$  are respectively

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx), \quad \sum_{n=1}^{\infty} b_n \sin(nx) \quad x \in [0, \pi].$$

Here are related tasks.

(i) Let  $f(x) = \frac{\pi}{2} - |x - \frac{\pi}{2}|$ ,  $x \in [0, \pi]$ .

(a) Compute  $b_n$  for all  $n \in \mathbb{N}$ . Write down the Fourier Series of Sines at each  $x \in [0, \pi]$ .

(b) Show that  $f$  is Lipschitz in  $[0, \pi]$ , and then deduce that the series from (a) converges to  $f(x)$  for all  $x \in [0, \pi]$ .

(c) Use (b) to deduce that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

(ii) Let  $f(x) = x^2 - \pi x$ ,  $x \in [0, \pi]$ .

(a) Compute  $b_n$  for all  $n \in \mathbb{N}$ . Write down the Fourier Series of Sines at each  $x \in [0, \pi]$ .

- (b) Show that  $f$  is Lipschitz in  $[0, \pi]$ , and then deduce that the series from (a) converges to  $f(x)$  for all  $x \in [0, \pi]$ .  
 (c) Use (b), evaluating at  $x = \pi/2$  to deduce that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

(iii) Let  $f(x) = (x - \frac{\pi}{2})^2$ ,  $x \in [0, \pi]$ .

- (a) Compute  $a_n$  for all  $n \in \mathbb{N} \cup \{0\}$ . Write down the Fourier Series of Cosines at each  $x \in [0, \pi]$ .  
 (b) Show that the series from (a) converges to  $f(x)$  for all  $x \in [0, \pi]$ .  
 (c) Use (b) to deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(iv) Let  $f(x) = x$ ,  $x \in [0, \pi]$ .

- (a) Compute  $a_n$  for all  $n \in \mathbb{N} \cup \{0\}$ . Write down the Fourier Series of Cosines at each  $x \in [0, \pi]$ .  
 (b) Show that the series from (a) converges to  $f(x)$  for all  $x \in [0, \pi]$ .  
 (c) Use (b) to deduce that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

(v) Let  $f(x) = \cosh x := \frac{e^x + e^{-x}}{2}$ ,  $x \in [0, \pi]$ .

- (a) Compute  $a_n$  for all  $n \in \mathbb{N} \cup \{0\}$ . Write down the Fourier Series of Cosines at each  $x \in [0, \pi]$ .  
 (b) Show that the series from (a) converges to  $f(x)$  for all  $x \in [0, \pi]$ .  
 (c) Use (b) to deduce that

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{\pi}{2 \tanh \pi} - \frac{1}{2}.$$

(vi) Let  $f(x) = \sinh x := \frac{e^x - e^{-x}}{2}$ ,  $x \in [0, \pi]$ .

- (a) Compute  $b_n$  for all  $n \in \mathbb{N}$ . Write down the Fourier Series of Sines at each  $x \in [0, \pi]$ .  
 (b) Show that the series from (a) converges to  $f(x)$  for all  $x \in [0, \pi]$ .

**Exercise 6.4.** For the functions  $f : [0, \pi] \rightarrow \mathbb{R}$  from Exercise 6.3(i),(ii), write down the Series Formula of a solution  $u(x, t)$  of the Heat Equation in  $[0, \pi]$  with initial data  $f$  :

$$(HEI) \equiv \begin{cases} \frac{\partial^2 u}{\partial x^2}(x, t) = \frac{\partial u}{\partial t}(x, t), & \text{if } (x, t) \in (0, \pi) \times (0, +\infty); \\ u(0, t) = u(\pi, t) = 0, & \text{if } t \in [0, +\infty); \\ u(x, 0) = f(x), & \text{if } x \in [0, \pi]. \end{cases}$$

**Exercise 6.5.** For the functions  $f : [0, \pi] \rightarrow \mathbb{R}$  from Exercise 6.3(i),(ii), write down the Series Formula of a solution  $u(x, t)$  of the Wave Equation in  $[0, \pi]$  with initial data  $f$  :

$$(WEI) \equiv \begin{cases} \frac{\partial^2 u}{\partial x^2}(x, t) = \frac{\partial^2 u}{\partial t^2}(x, t), & \text{if } (x, t) \in (0, \pi) \times (0, +\infty); \\ u(0, t) = u(\pi, t) = 0, & \text{if } t \in [0, +\infty); \\ \frac{\partial u}{\partial t}(x, 0) = 0, & \text{if } x \in [0, \pi]; \\ u(x, 0) = f(x), & \text{if } x \in [0, \pi]. \end{cases}$$