SOLUTIONS TO THE EXAM MA2106-27 NOVEMBER 2024

Exercise 1 (1 point)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be the function

$$f(z) = \cos(\overline{z}), \quad z \in \mathbb{C}.$$

Find $\mathrm{Re}(f)(x, y)$ and $\mathrm{Im}(f)(x, y)$, that is, the real and imaginary parts of f.

Solution: We look for functions $u(x, y), v(x, y)$ in $\mathbb{R}^2$, for which

$$f(x + iy) = u(x, y) + iv(x, y), \quad (x, y) \in \mathbb{R}^2.$$

First, for $z = x + iy$, we write

$$\overline{z} = x - iy,$$

and the definition of the complex cosine gives

$$\begin{aligned}
\cos(\overline{z}) &= \frac{e^{i\overline{z}} + e^{-i\overline{z}}}{2} = \frac{e^{i(x-iy)} + e^{-i(x-iy)}}{2} = \frac{e^{ix}e^{y} + e^{-ix}e^{-y}}{2} \\
&= \frac{e^{y}(\cos x + i \sin x) + e^{-y}(\cos(-x) + i\sin(-x))}{2} \\
&= \frac{e^{y}(\cos x + i \sin x) + e^{-y}(\cos x - i \sin x)}{2} \\
&= \frac{(e^{y} + e^{-y})\cos x + i\,(e^{y} - e^{-y})\sin x}{2}.
\end{aligned}$$

We thus have that

$$\begin{aligned}
\mathrm{Re}(f)(x, y) = u(x, y) &= \frac{(e^{y} + e^{-y})\cos x}{2}, \\
\mathrm{Im}(f)(x, y) = v(x, y) &= \frac{(e^{y} - e^{-y})\sin x}{2}.
\end{aligned}$$

□

Exercise 2 (1'5 points)**.** Let u be the function $u : \mathbb{R}^2 \to \mathbb{R}$

$$u(x, y) = x^2 - y^2 + 2x + e^x \sin y, \quad (x, y) \in \mathbb{R}^2.$$

Show that u is harmonic in $\mathbb{R}^2$ by veryfing the Laplace Equation $\Delta u(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$. Then, find the harmonic conjugate $v : \mathbb{R}^2 \to \mathbb{R}$ of u that satisfies $v(0, 0) = 0$.

Solution: We prove the harmonicity via the Laplace Equation $\Delta u = 0$. Indeed,

$$\begin{aligned}
\frac{\partial u}{\partial x}(x, y) &= 2x + 2 + e^x \sin y, \quad \frac{\partial^2 u}{\partial x^2}(x, y) = 2 + e^x \sin y, \\
\frac{\partial u}{\partial y}(x, y) &= -2y + e^x \cos y, \quad \frac{\partial^2 u}{\partial y^2}(x, y) = -2 - e^x \sin y.
\end{aligned}$$

and so

$$\Delta u(x, y) = \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 2 + e^x \sin y - 2 - e^x \sin y = 0.$$

Thus, u is harmonic in $\mathbb{R}^2$. Now, recall that a harmonic conjugate is a function $v : \mathbb{R}^2 \to \mathbb{R}$ so that $u + iv$ is holomorphic. To find such function v, we solve the following system of (Cauchy-Riemann) Differential Equations:

$$\frac{\partial v}{\partial y}(x,y) = \frac{\partial u}{\partial x}(x,y) = 2x + 2 + e^x \sin y$$
$$\frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y) = 2y - e^x \cos y.$$

Integrating with respect to y in the first one, we see that

$$v(x,y) = \int (2x + 2 + e^x \sin y)\,\mathrm{d}y = 2xy + 2y - e^x \cos y + \varphi(x),$$

for some differentiable $\varphi : \mathbb{R} \to \mathbb{R}$. Plugging this into the second equation leads to

$$2y - e^x \cos y + \varphi'(x) = 2y - e^x \cos y.$$

And this implies that $\varphi'(x) = 0$ and therefore $\varphi(x) = K$, with $K \in \mathbb{R}$ cosntant. The expression for v becomes

$$v(x,y) = 2xy + 2y - e^x \cos y + K, \quad K \in \mathbb{R}.$$

Since we are interested in the v that satisfies $v(0,0) = 0$, we see that necessarily $K = 1$, whence

$$v(x,y) = 2xy + 2y - e^x \cos y + 1.$$

□

Exercise 3 (1 point)**.** Use the Cauchy Integral Formula to evaluate the complex path-integral:

$$\int_{\partial D(0,2)} \frac{z^2 e^z}{(z-1)(z-4)}\,\mathrm{d}z;$$

where $\partial D(0,2)$ is the circle centered at 0 and with radius 2 traveled once and counterclockwise.

Solution: We can use many versions of the Cauchy Integral Formula. For example, the Local Cauchy Integral Formula in a circle (Corollary 4.28, p. 94 from the Lecture Notes):

Theorem. *Let $\Omega \subset \mathbb{C}$ be an open set, $\overline{D}(z_0, r) \subset \Omega$, and let $f : \Omega \to \mathbb{C}$ be holomorphic. Then,*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(w)}{w - z}\,\mathrm{d}w, \quad \textit{for all} \quad z \in D(z_0, r);$$

where $\partial D(z_0, r)$ is traveled once and counterclockwise.

In our exercise, consider $\Omega = D(0,3)$, and note that $\overline{D}(0,2) \subset \Omega$. Moreover, the function

$$f(z) = \frac{z^2 e^z}{z - 4}, \quad z \in \Omega,$$

is holomorphic in Ω, since the potential problem is at 4, not contained in Ω. The Cauchy Integral Formula above says that then

$$\int_{\partial D(0,2)} \frac{z^2 e^z}{(z-1)(z-4)}\,\mathrm{d}z = \int_{\partial D(0,2)} \frac{f(z)}{z-1}\,\mathrm{d}z = 2\pi i f(1) = -\frac{2e\pi i}{3}.$$

□

Exercise 4 (3 points)**.** Use the Cauchy Residues Theorem to evaluate:

(a) The complex path-integral:
$$\int_{\partial D(0,5)} \frac{\cos z}{e^z - 1}\,\mathrm{d}z;$$
where $\partial D(0,5)$ is the circle centered at 0 and with radius 5 traveled once and counterclockwise.

(b) The real integral:
$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{5 - 4\cos\theta}.$$

(c) The principal value of the integral:
$$\mathrm{pv}\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2 + a^2}\,\mathrm{d}x,$$
where $a \in \mathbb{R}$ and $a > 0$.

Proof. The following proposition is instrumental for the calculus of residues.

Theorem. *Let $g, h : \Omega \to \mathbb{C}$ holomorphic functions and $z_0 \in \Omega$. Assume that:*

- *$g(z_0) \neq 0$, and*
- *$h(z_0) = 0$ and $h'(z_0) \neq 0$.*

Then $f := \dfrac{g}{h}$ has a simple pole (pole of order 1) at z_0 and
$$\mathrm{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}.$$

(a) Recall the Cauchy Residues Theorem:

Theorem (Cauchy Residues Theorem). *Let $f : \Omega \setminus \{z_1, \dots, z_N\} \to \mathbb{C}$ be a holomorphic function in Ω except at N distinct points $\{z_1, \dots, z_N\} \subset \Omega$, at which f has isolated singularities. Let $\gamma : [a,b] \to \Omega \setminus \{z_1, \dots, z_N\}$ be a closed and piecewise C^1-path with $W(\gamma, z) = 0$ for all $z \notin \Omega$. Then, we have*
$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{N} \mathrm{Res}(f, z_k) W(\gamma, z_k).$$

Our function
$$f(z) = \frac{\cos z}{e^z - 1}$$
has singularities at those z such that $e^z = 1$, that is, if and only if $z = 2k\pi i$, for $k \in \mathbb{Z}$. But among those, only 0 is contained in the inside of the circle $\partial D(0,5)$, as, for $k \in \mathbb{Z} \setminus \{0\}$, one has
$$|2k\pi i| \geq 2\pi > 6 > 5.$$
So, by the Cauchy Residues Theorem above, we have
$$\int_{\partial D(0,5)} \frac{\cos z}{e^z - 1}\,\mathrm{d}z = 2\pi i\, W(\partial D(0,5), 0)\, \mathrm{Res}(f, 0).$$
Note that $W(\partial D(0,5), 0) = 1$ because $\partial D(0,5)$ travels precisely one time and in the positive orientation around 0. Therefore,
$$\int_{\partial D(0,5)} \frac{\cos z}{e^z - 1}\,\mathrm{d}z = 2\pi i\, \mathrm{Res}(f, 0).$$

Let's calculate the residue. We write

$$f(z) = \frac{g(z)}{h(z)}; \quad g(z) = \cos z, \quad h(z) = e^z - 1, \quad h'(z) = e^z.$$

When $z = 0$, we have that

$$g(0) = 1, \quad h(0) = 0, \quad h'(0) = e^0 = 1.$$

By the Proposition, we have that

$$\mathrm{Res}(f, 0) = \frac{g(0)}{h'(0)} = 1$$

We can conclude that

$$\int_{\partial D(0,5)} \frac{\cos z}{e^z - 1} \, \mathrm{d}z = 2\pi i.$$

(b) The following theorem takes care of integrals in $[0, 2\pi]$ of bounded rational-trigonometric functions:

Theorem (Evaluation of Trigonometric Integrals)**.** *Let $R(u, v)$ be a rational function of two variables such that the function $[0, 2\pi] \ni \theta \mapsto R(\cos\theta, \sin\theta)$ is bounded in $[0, 2\pi]$. Consider the function*

$$f(z) := \frac{1}{iz} R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right),$$

and denote $\mathrm{Poles}(f) := \{z \in \mathbb{C} : f \text{ has a pole at } z\}$ *and $\mathbb{D} = D(0, 1)$ the open unit disk. Then,*

$$\int_0^{2\pi} R(\cos\theta, \sin\theta) \, \mathrm{d}\theta = 2\pi i \sum_{z \in \mathbb{D} \cap \mathrm{Poles}(f)} \mathrm{Res}(f, z).$$

In our exercise, we have that

$$R(u, v) = \frac{1}{5 - 4u}.$$

And the function f is

$$f(z) = \frac{1}{iz} \cdot \frac{1}{5 - 4\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right)} = \frac{1}{iz} \cdot \frac{1}{5 - 2\left(z + \frac{1}{z}\right)} = \frac{1}{i} \cdot \frac{1}{-2z^2 + 5z - 2} = \frac{i}{2z^2 - 5z + 2}.$$

By the theorem,

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{5 - 4\cos\theta} = 2\pi i \sum_{z \in \mathbb{D} \cap \mathrm{Poles}(f)} \mathrm{Res}(f, z).$$

Now we need to indentify the singularities of f and select those that are contained in $\mathbb{D}$. The zeros of $2z^2 - 5z + 2$ are

$$z = 1/2, \ 2.$$

Thus we need to confirm that f has a pole in $1/2$ and calculate the corresponding residue. The other singularity 2 is not in $\mathbb{D}$, so we ignore it. But observe that

$$f(z) = \frac{g(z)}{h(z)}, \quad g(z) = i, \quad h(z) = 2z^2 - 5z + 2, \quad h'(z) = 4z - 5.$$

We have that $g(1/2) = i \neq 0$, $h(1/2) = 0$, $h'(1/2) = -3 \neq 0$. By the proposition,

$$\mathrm{Res}(f, 1/2) = \frac{-i}{3}.$$

We can conclude

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{5-4\cos\theta} = 2\pi i \sum_{z\in\mathbb{D}\cap\mathrm{Poles}(f)} \mathrm{Res}(f,z) = 2\pi i\,\mathrm{Res}(f,1/2) = 2\pi i\left(\frac{-i}{3}\right) = \frac{2\pi}{3}.$$

(c) For this type of integral, we use:

Theorem. *Denote $\mathbb{H} := \{z\in\Omega : \mathrm{Im}(z)\geq 0\}$ the upper half-plane and let $\Omega\subset\mathbb{C}$ be open with $\mathbb{H}\subset\Omega$. Let f be a function with the following conditions*

- *f is holomorphic in Ω except at finitely many singularities $z_1,\dots,z_N\in\Omega$.*
- *$z_k\notin\mathbb{R}$ (that is, $\mathrm{Im}(z_k)\neq 0$) for all $k=1,\dots,N$.*
- *There are constants $M, R_0>0$, $p>0$ such that $|f(z)|\leq\frac{M}{|z|^p}$ for all $z\in\Omega$ with $|z|>R_0$.*

Then, for all $b>0$, if $g(z) := f(z)e^{ibz}$, $z\in\Omega$, we have

$$\mathrm{pv}\int_{-\infty}^{+\infty} f(x)e^{ibx}\,\mathrm{d}x = 2\pi i\sum_{\{k\,:\,\mathrm{Im}(z_k)>0\}}\mathrm{Res}(g,z_k).$$

In our problem,

$$f(z) = \frac{1}{z^2+a^2},\quad g(z) = e^{iz}f(z).$$

The singularities of f are those $z\in\mathbb{C}$ such that $z^2=-a^2$, that is

$$z = ai,\quad z=-ai.$$

So, f has finitely many singularities, none of them in the real line, as $a>0$, by the assumption. Also, f is quotient of the polynomials 1 and z^2+a^2; where $\deg(z^2+a^2)\geq 1+\deg(1)$. Thus, the theorem applies for our problem, and so

$$\mathrm{pv}\int_{-\infty}^{+\infty} f(x)e^{ix}\,\mathrm{d}x = 2\pi i\sum_{\{k\,:\,\mathrm{Im}(z_k)>0\}}\mathrm{Res}(g,z_k).$$

Moreover, we only need to examine those singularities whose imaginary part is positive. Therefore, we only need to look at ai, whereas $-ai$ can be ignored. And we now calculate the residues of g (not of f!) at those points. Observe that

$$g(z) = \frac{\varphi(z)}{\psi(z)},\quad \varphi(z)=e^{iz},\quad \psi(z)=z^2+a^2,\quad \psi'(z)=2z.$$

For our singularity ai, we have

$$\varphi(ai) = e^{i(ia)} = e^{-a}\neq 0,\quad \psi(ai)=0,\quad \psi'(ai)=2ai\neq 0.$$

Our proposition for the calculus of residues tells us that g is a pole of order 1 at ai, with

$$\mathrm{Res}(g,ai) = \frac{\varphi(ai)}{\psi'(ai)} = \frac{e^{-a}}{2ai}.$$

Then we can conclude

$$\mathrm{pv}\int_{-\infty}^{+\infty} f(x)e^{ix}\,\mathrm{d}x = 2\pi i\,\mathrm{Res}(g,ai) = 2\pi i\left(\frac{e^{-a}}{2ai}\right) = \frac{\pi e^{-a}}{a}.$$

$\square$

Exercise 5 (1 point)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function in $\mathbb{C}$ for which there are constants $m \in \mathbb{N}$, $r_0 > 0$, and $C > 0$ such that

$$|f(z)| \le C|z|^m \log(1 + |z|), \quad \text{whenever} \quad |z| \ge r_0.$$

Prove that f is a polynomial of degree at most m.

Suggestion: Recall the Cauchy Estimates for the derivatives $f^{(k)}$ of f.

Solution: Recall the Cauchy Estimates for the derivatives of holomorphic functions in $\mathbb{C}$:

Theorem. *Let $g : \mathbb{C} \to \mathbb{C}$ be holomorphic. Then, for every $n \in \mathbb{N}$, $r > 0$, and $z \in \mathbb{C}$: one has*

$$\left|g^{(n)}(z)\right| \le \frac{n!}{r^n} \sup\{|g(w)| \,:\, |w - z| = r\}.$$

For our problem, consider $n = m + 1$ and apply the Cauchy Estimates for $f^{(m+1)}$:

$$\left|f^{(m+1)}(z)\right| \le \frac{(m+1)!}{r^{m+1}} \sup\{|f(w)| \,:\, |w - z|\}, \quad \text{for all} \quad z \in \mathbb{C}, r > 0.$$

Now, we fix $z \in \mathbb{C}$, and let $r \ge r_0 + |z|$, where r_0 is that from the assumption. Note that then $|w - z| = r$ implies, by the triangle inequality,

$$|w| \ge |w - z| - |z| = r - |z| \ge r_0, \quad \text{and} \quad |w| \le |w - z| + |z| = r + |z|.$$

Applying first the assumption and then the latter estimate, we get

$$|f(w)| \le C|w|^m \log(1 + |w|) \le C(r + |z|)^m \log(1 + r + |z|), \quad \text{whenever} \quad |w - z| = r.$$

Inserting this back into the estimate for $\left|f^{(m+1)}(z)\right|$, we derive

$$\left|f^{(m+1)}(z)\right| \le \frac{(m+1)!C(r + |z|)^m \log(1 + r + |z|)}{r^{m+1}} = (m+1)!\,C\left(1 + \frac{|z|}{r}\right)^m \frac{\log(1 + r + |z|)}{r}.$$

Letting $r \to \infty$, (with $r \ge r_0$, and taking into account that z is fixed), the last term goes to 0; implying that

$$f^{(m+1)}(z) = 0.$$

Since $z \in \mathbb{C}$ was arbitrary, we deduce that $f^{(m+1)} \equiv 0$ in $\mathbb{C}$. This shows that f must be a polynomial of degree at most m. □

Exercise 6 (1 point)**.** Let $D(0, 1)$ be the open disk centered at 0 and with radius 1. Let $f : D(0, 1) \to \mathbb{C}$ be a holomorphic function in $D(0, 1)$ satisfying:

$$f\left(\frac{1}{\sqrt{n}}\right) = \frac{n}{n - i}, \quad \text{for all} \quad n \in \mathbb{N}, n \ge 2.$$

Find the explicit formula of $f(z)$, for all $z \in D(0, 1)$, and calculate the derivatives $f^{(k)}(0)$ of f at 0, for all $k \in \mathbb{N}$.

Suggestion: Use the Second Identity Principle for Holomorphic Functions.

Solution: Recall the Second Identity Principle For Holomorphic Maps:

Theorem. *Let $\Omega \subset \mathbb{C}$ be open and connected, and $f, g : \Omega \to \mathbb{C}$ two holomorphic functions such that there are $z_0 \in \Omega$ and a sequence $\{z_k\}_k \subset \Omega \setminus \{z_0\}$ such that $\lim\limits_{k\to\infty} z_k = z_0$ and $f(z_k) = g(z_k)$ for all $k \in \mathbb{N}$. Then $f = g$ on Ω.*

In our problem, note that

$$f\left(\frac{1}{\sqrt{n}}\right) = \frac{n}{n-i} = \frac{1}{1 - i \cdot \frac{1}{n}} = \frac{1}{1 - i \cdot \left(\frac{1}{\sqrt{n}}\right)^2}, \quad n \geq 2.$$

So we consider the function $g : D(0,1) \to \mathbb{C}$ given by

$$g(z) = \frac{1}{1 - iz^2}, \quad z \in D(0,1).$$

This is a holomorphic function, since $|iz^2| = |z|^2 < 1$ whenever $z \in D(0,1)$. And we have seen that $f\left(\frac{1}{\sqrt{n}}\right) = g\left(\frac{1}{\sqrt{n}}\right)$ for all $n \in \mathbb{N}$, $n \geq 2$. The sequence $\left\{\frac{1}{\sqrt{n}}\right\}_{n\geq 2}$ is contained in $D(0,1)$, and converges to $0 \in D(0,1)$. Moreover, $\frac{1}{\sqrt{n}} \neq 0$ for all $n \geq 2$. Since both g and f are holomorphic, we may conclude that

$$f(z) = g(z) = \frac{1}{1 - iz^2}, \quad \text{for all} \quad z \in D(0,1).$$

To calculate the derivatives $f^{(k)}(0)$, we can write f as the power series

$$f(z) = \frac{1}{1 - iz^2} = \sum_{n=0}^{\infty} (iz^2)^n = \sum_{n=0}^{\infty} i^n z^{2n}, \quad z \in D(0,1).$$

The radius of convergence of this series is

$$\left(\limsup_{n\to\infty} |i^n|^{1/2n}\right)^{-1} = 1.$$

So, for example by Corollary 3.21, p. 68 Lecture Notes, the terms of the series above are precisely the terms of the Taylor Series of f centered at 0, that is,

$$f(z) = \sum_{n=0}^{\infty} i^n z^{2n} = \sum_{n=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k,$$

and

$$f^{(k)}(0) = \begin{cases} k!\, i^{k/2} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

$\square$

Exercise 7 (1.5 points)**.** Let $f : [0,\pi] \to \mathbb{R}$ be the function defined by

$$f(x) = x^2 - \pi x, \quad \text{for all} \quad x \in [0,\pi].$$

(a) Calculate the Sine-Fourier Coefficients of f in $[0,\pi]$, that is,

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(t) \sin(nt)\, \mathrm{d}t, \quad n \in \mathbb{N}.$$

Then write down the Sine-Fourier Series $S(f)(x)$ of f for all $x \in [0,\pi]$.

(b) Show that f is Lipschitz and then conclude that the Fourier series $S(f)(x)$ of f converges to $f(x)$ for all $x \in [0,\pi]$.

(c) If f is the function above, consider the Heat Equation in $[0,\pi]$ with boundary conditions:

$$(\mathrm{P}) \equiv \begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,t) = \dfrac{\partial u}{\partial t}(x,t); & \text{if} \quad (x,t) \in (0,\pi) \times (0,+\infty) \\ u(0,t) = u(\pi,t) = 0 & \text{if} \quad t \in [0,+\infty) \\ u(x,0) = f(x) & \text{if} \quad x \in [0,\pi]. \end{cases}$$

Write down a solution $u(x,t)$ of (P) as a series of functions.

Clarification: *You do NOT need to justify why this series $u(x,t)$ is a solution of* (P).

Solution:

(a) For every $n \in \mathbb{N}$, we use integration by parts:

$$\int_0^\pi (t^2 - \pi t)\sin(nt)\,\mathrm{d}t = \left[-(t^2-\pi t)\frac{\cos(nt)}{n}\right]_{t=0}^{t=\pi} - \int_0^\pi (2t-\pi)\left(-\frac{\cos(nt)}{n}\right)\mathrm{d}t = \int_0^\pi (2t-\pi)\frac{\cos(nt)}{n}\,\mathrm{d}t$$

$$= \left[-(2t-\pi)\frac{\sin(nt)}{n^2}\right]_{t=0}^{t=\pi} - 2\int_0^\pi \frac{\sin(nt)}{n^2}\,\mathrm{d}t = \left[2\frac{\cos(nt)}{n^3}\right]_{t=0}^{t=\pi} = 2\frac{(-1)^n - 1}{n^3}.$$

Then,

$$b_n = \frac{2}{\pi}\int_0^\pi (t^2-\pi t)\sin(nt)\,\mathrm{d}t = \frac{4}{\pi}\frac{(-1)^n-1}{n^3} = \begin{cases} -\frac{8}{\pi n^3} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

The Fourier Series of Sines of f in $[0,\pi]$ is

$$S(f)(x) = \sum_{n=1}^{\infty} b_n \sin(nx) = -\frac{8}{\pi}\sum_{n=0}^{\infty}\frac{\sin((2n+1)x)}{(2n+1)^3}, \quad x \in [0,\pi].$$

(b) We need to find $C > 0$ so that

$$|f(x) - f(y)| \le C|x-y|, \quad x,y \in [0,\pi].$$

Indeed, for $x, y \in [0,\pi]$:

$$|f(x)-f(y)| = |x^2 - \pi x - y^2 + \pi y| \le |x^2 - y^2| + \pi|x-y|$$
$$= |x+y||x-y| + \pi|x-y| = (|x+y| + \pi)\,|x-y| \le 3\pi|x-y|.$$

Thus f is Lipschitz in $[0,\pi]$.

And we know that then $S(f)(x) = f(x)$ for all $x \in [0,\pi]$; where

$$S(f)(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

(c) A solution $u(x,t)$ to the problem (P) in series form is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t}\sin(nx) = -\frac{8}{\pi}\sum_{n=0}^{\infty}\frac{\sin((2n+1)x)e^{-(2n+1)^2 t}}{(2n+1)^3},$$

for all $(x,t) \in [0,\pi] \times [0,+\infty)$. $\square$