

**Exercise 1** (1 point). Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be the function

$$f(z) = \cos(\bar{z}), \quad z \in \mathbb{C}.$$

Find  $\operatorname{Re}(f)(x, y)$  and  $\operatorname{Im}(f)(x, y)$ , that is, the real and imaginary parts of  $f$ .

**Exercise 2** (1'5 points). Let  $u$  be the function  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$u(x, y) = x^2 - y^2 + 2x + e^x \sin y, \quad (x, y) \in \mathbb{R}^2.$$

Show that  $u$  is harmonic in  $\mathbb{R}^2$  by verifying the Laplace Equation  $\Delta u(x, y) = 0$  for all  $(x, y) \in \mathbb{R}^2$ . Then, find the harmonic conjugate  $v : \mathbb{R}^2 \rightarrow \mathbb{R}$  of  $u$  that satisfies  $v(0, 0) = 0$ .

**Exercise 3** (1 point). Use the Cauchy Integral Formula to evaluate the complex path-integral:

$$\int_{\partial D(0,2)} \frac{z^2 e^z}{(z-1)(z-4)} dz;$$

where  $\partial D(0, 2)$  is the circle centered at 0 and with radius 2 traveled once and counterclockwise.

**Exercise 4** (3 points). Use the Cauchy Residues Theorem to evaluate:

(a) The complex path-integral:

$$\int_{\partial D(0,5)} \frac{\cos z}{e^z - 1} dz;$$

where  $\partial D(0, 5)$  is the circle centered at 0 and with radius 5 traveled once and counterclockwise.

(b) The real integral:

$$\int_0^{2\pi} \frac{d\theta}{5 - 4 \cos \theta}.$$

(c) The principal value of the integral:

$$\operatorname{pv} \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2 + a^2} dx,$$

where  $a \in \mathbb{R}$  and  $a > 0$ .

**Exercise 5** (1 point). Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a holomorphic function in  $\mathbb{C}$  for which there are constants  $m \in \mathbb{N}$ ,  $r_0 > 0$ , and  $C > 0$  such that

$$|f(z)| \leq C|z|^m \log(1 + |z|), \quad \text{whenever } |z| \geq r_0.$$

Prove that  $f$  is a polynomial of degree at most  $m$ .

*Suggestion: Recall the Cauchy Estimates for the derivatives  $f^{(k)}$  of  $f$ .*

**Exercise 6** (1 point). Let  $D(0, 1)$  be the open disk centered at 0 and with radius 1. Let  $f : D(0, 1) \rightarrow \mathbb{C}$  be a holomorphic function in  $D(0, 1)$  satisfying:

$$f\left(\frac{1}{\sqrt{n}}\right) = \frac{n}{n-i}, \quad \text{for all } n \in \mathbb{N}, n \geq 2.$$

Find the explicit formula of  $f(z)$ , for all  $z \in D(0, 1)$ , and calculate the derivatives  $f^{(k)}(0)$  of  $f$  at 0, for all  $k \in \mathbb{N}$ .

*Suggestion: Use the Second Identity Principle for Holomorphic Functions.*

**Exercise 7** (1.5 points). Let  $f : [0, \pi] \rightarrow \mathbb{R}$  be the function defined by

$$f(x) = x^2 - \pi x, \quad \text{for all } x \in [0, \pi].$$

(a) Calculate the Sine-Fourier Coefficients of  $f$  in  $[0, \pi]$ , that is,

$$b_n = \frac{2}{\pi} \int_0^\pi f(t) \sin(nt) dt, \quad n \in \mathbb{N}.$$

Then write down the Sine-Fourier Series  $S(f)(x)$  of  $f$  for all  $x \in [0, \pi]$ .

(b) Show that  $f$  is Lipschitz and then conclude that the Fourier series  $S(f)(x)$  of  $f$  converges to  $f(x)$  for all  $x \in [0, \pi]$ .

(c) If  $f$  is the function above, consider the Heat Equation in  $[0, \pi]$  with boundary conditions:

$$(P) \equiv \begin{cases} \frac{\partial^2 u}{\partial x^2}(x, t) = \frac{\partial u}{\partial t}(x, t); & \text{if } (x, t) \in (0, \pi) \times (0, +\infty) \\ u(0, t) = u(\pi, t) = 0 & \text{if } t \in [0, +\infty) \\ u(x, 0) = f(x) & \text{if } x \in [0, \pi]. \end{cases}$$

Write down a solution  $u(x, t)$  of (P) as a series of functions.

**Clarification:** You do NOT need to justify why this series  $u(x, t)$  is a solution of (P).