SOLUTIONS TO THE 2^{nd} EXAM DRILL-MA2106. 19 NOVEMBER 2024

Exercise 1 (1 point)**.** For the function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = \sin\left(z^2\right), \quad z \in \mathbb{C},$$

find $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$, that is, the real and imaginary parts of f.

Solution: We look for functions $u(x,y), v(x,y)$ in $\mathbb{R}^2$, for which

$$f(x+iy) = u(x,y) + iv(x,y), \quad (x,y) \in \mathbb{R}^2.$$

First, for $z = x+iy$, we write

$$z^2 = x^2 - y^2 + 2xyi.$$

Now, the definition of the complex sine gives

$$\begin{aligned}
\sin\left(z^2\right) &= \frac{e^{iz^2} - e^{-iz^2}}{2i} = \frac{e^{i\left(x^2-y^2+2xyi\right)} - e^{-i\left(x^2-y^2+2xyi\right)}}{2i} = \frac{e^{-2xy}e^{(x^2-y^2)i} - e^{2xy}e^{-i(x^2-y^2)}}{2i} \\
&= \frac{e^{-2xy}\left(\cos(x^2-y^2) + i\sin(x^2-y^2)\right) - e^{2xy}\left(\cos(x^2-y^2) - i\sin(x^2-y^2)\right)}{2i} \\
&= \frac{e^{-2xy}\cos(x^2-y^2) - e^{2xy}\cos(x^2-y^2)}{2i} + \frac{ie^{-2xy}\sin(x^2-y^2) + ie^{2xy}\sin(x^2-y^2)}{2i} \\
&= -i\frac{e^{-2xy}\cos(x^2-y^2) - e^{2xy}\cos(x^2-y^2)}{2} + \frac{e^{-2xy}\sin(x^2-y^2) + e^{2xy}\sin(x^2-y^2)}{2} \\
&= i\frac{e^{2xy}\cos(x^2-y^2) - e^{-2xy}\cos(x^2-y^2)}{2} + \frac{e^{-2xy}\sin(x^2-y^2) + e^{2xy}\sin(x^2-y^2)}{2}
\end{aligned}$$

We thus have that

$$\begin{aligned}
\mathrm{Re}(f)(x,y) = u(x,y) &= \frac{e^{-2xy}\sin(x^2-y^2) + e^{2xy}\sin(x^2-y^2)}{2}, \\
\mathrm{Im}(f)(x,y) = v(x,y) &= \frac{e^{2xy}\cos(x^2-y^2) - e^{-2xy}\cos(x^2-y^2)}{2}.
\end{aligned}$$

□

Exercise 2 (1'5 points)**.** For the function $u : \mathbb{R}^2 \to \mathbb{R}$ given by

$$u(x,y) = 3x^2y + 3x^2 - y^3 - 3y^2 - 3y + e^x\cos y, \quad (x,y) \in \mathbb{R}^2,$$

prove that u is harmonic in $\mathbb{R}^2$ by veryfing the laplace equation $\Delta u(x,y) = 0$, for all $(x,y) \in \mathbb{R}^2$. Then, find the harmonic conjugate v of u that satisfies $v(0,0) = 1$.

Solution: We prove the harmonicity via the Laplace Equation $\Delta u = 0$. Indeed,

$$\begin{aligned}
&\frac{\partial u}{\partial x}(x,y) = 6xy + 6x + e^x\cos y, \quad \frac{\partial^2 u}{\partial x^2}(x,y) = 6y + 6 + e^x\cos y, \\
&\frac{\partial u}{\partial y}(x,y) = 3x^2 - 3y^2 - 6y - 3 - e^x\sin y, \quad \frac{\partial^2 u}{\partial y^2}(x,y) = -6y - 6 - e^x\cos y.
\end{aligned}$$

and so

$$\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 6y + 6 + e^x \cos y - 6y - 6 - e^x \cos y = 0.$$

Thus, u is harmonic in $\mathbb{R}^2$. Now, recall that a harmonic conjugate is a function $v : \mathbb{R}^2 \to \mathbb{R}$ so that $u + iv$ is holomorphic. To find such function v, we solve the following system of (Cauchy-Riemann) Differential Equations:

$$\frac{\partial v}{\partial y}(x,y) = \frac{\partial u}{\partial x}(x,y) = 6xy + 6x + e^x \cos y$$
$$\frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y) = -3x^2 + 3y^2 + 6y + 3 + e^x \sin y.$$

Integrating with respect to y in the first one, we see that

$$v(x,y) = \int (6xy + 6x + e^x \cos y)\, \mathrm{d}y = 3xy^2 + 6xy + e^x \sin y + \varphi(x),$$

for some differentiable $\varphi : \mathbb{R} \to \mathbb{R}$. Plugging this into the second equation leads to

$$3y^2 + 6y + e^x \sin y + \varphi'(x) = -3x^2 + 3y^2 + 6y + 3 + e^x \sin y.$$

And this implies that $\varphi'(x) = -3x^2 + 3$ and therefore $\varphi(x) = -x^3 + 3x + K$, with $K \in \mathbb{R}$. Now that we know the expression for $\varphi(x)$ we use it in the expression for v :

$$v(x,y) = 3xy^2 + 6xy + e^x \sin y - x^3 + 3x + K, \quad K \in \mathbb{R}.$$

Since we are interested in the v that satisfies $v(0,0) = 1$, we see that necessarily $K = 1$, whence

$$v(x,y) = 3xy^2 + 6xy + e^x \sin y - x^3 + 3x + 1.$$

□

Exercise 3 (1 point)**.** Use the Cauchy Integral Formula to evaluate the complex path-integral:

$$\int_{\partial D(0,1)} \frac{\sin(z^2)}{\left(z + \frac{3}{2}\right)^2 \left(z - \frac{2}{3}\right)} \,\mathrm{d}z;$$

where $\partial D(0,1)$ is traveled once and counterclockwise.

<u>Solution</u>: We can use, for example, the Local Cauchy Integral Formula in a circle (Corollary 4.28, p. 94 from the Lecture Notes:

Theorem (Local Cauchy Integral Formula)**.** *Let $\Omega \subset \mathbb{C}$ be an open set, $\overline{D}(z_0, r) \subset \Omega$, and let $f : \Omega \to \mathbb{C}$ be holomorphic. Then,*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(w)}{w - z} \,\mathrm{d}w, \quad \textit{for all} \quad z \in D(z_0, r);$$

where $\partial D(z_0, r)$ is traveled once and counterclockwise.

In our exercise, consider $\Omega = D(0, 1 + \frac{1}{4})$, and note that $\overline{D}(0,1) \subset \Omega$. Moreover, the function

$$f(z) = \frac{\sin(z^2)}{\left(z + \frac{3}{2}\right)^2}, \quad z \in \Omega,$$

is holomorphic in Ω, since the potential problem is at $-3/2$, not contained in Ω. The Local Cauchy Integral Formula says that then

$$\int_{\partial D(0,1)} \frac{\sin(z^2)}{\left(z + \frac{3}{2}\right)^2 \left(z - \frac{2}{3}\right)} \,\mathrm{d}z = \int_{\partial D(0,1)} \frac{f(z)}{z - \frac{2}{3}} \,\mathrm{d}z = 2\pi i f\left(\frac{2}{3}\right) = 2\pi i f(-2) = 2\pi i \frac{\sin\left(\frac{4}{9}\right)}{\left(\frac{2}{3} + \frac{3}{2}\right)^2} = 72\pi i \frac{\sin\left(\frac{4}{9}\right)}{169}.$$

$\square$

Exercise 4 (3 points)**.** Use the Cauchy Residues Theorem to evaluate:

(a) The complex path-integral:

$$\int_{\partial D(0,1/2)} \frac{(e^z+1)\cos z}{\sin(\pi z)}\,\mathrm{d}z;$$

where $\partial D(0,1/2)$ is the circle centered at 0 and with radius $1/2$, traveled once and counter-clockwise.

(b) The real integral:

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{13+12\cos\theta}.$$

(c) The principal value of the real integral:

$$\mathrm{pv}\int_{-\infty}^{+\infty} \frac{x^2\cos(ax)}{x^4+1}\,\mathrm{d}x,$$

for a $a\in\mathbb{R}$, $a>0$.

Proof. The following proposition is instrumental for the calculus of residues.

Theorem. *Let $g,h:\Omega\to\mathbb{C}$ holomorphic functions and $z_0\in\Omega$. Assume that:*

- *$g(z_0)\neq 0$, and*
- *$h(z_0)=0$ and $h'(z_0)\neq 0$.*

Then $f:=\dfrac{g}{h}$ has a simple pole (pole of order 1) at z_0 and

$$\mathrm{Res}(f,z_0)=\frac{g(z_0)}{h'(z_0)}.$$

(a) Recall the Cauchy Residues Theorem:

Theorem (Cauchy Residues Theorem)**.** *Let $f:\Omega\setminus\{z_1,\dots,z_N\}\to\mathbb{C}$ be a holomorphic function in Ω except at N distinct points $\{z_1,\dots,z_N\}\subset\Omega$, at which f has isolated singularities. Let $\gamma:[a,b]\to\Omega\setminus\{z_1,\dots,z_N\}$ be a closed and piecewise C^1-path with $W(\gamma,z)=0$ for all $z\notin\Omega$. Then, we have*

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i\sum_{k=1}^N \mathrm{Res}(f,z_k)W(\gamma,z_k).$$

Our concrete function $f(z)=\frac{(e^z+1)\cos z}{\sin(\pi z)}$ has singularities at those z such that $\sin(\pi z)=0$. Recall that $\sin w=0$ if and only $w=k\pi$ for some $k\in\mathbb{Z}$; this follows by noticing that

$$\sin w=0 \iff \frac{e^{iw}-e^{-iw}}{2i}=0 \iff e^{2iw}=1 \iff 2iw\in 2\pi i\mathbb{Z} \iff w\in\pi\mathbb{Z}.$$

But among those singularities $\{k\pi\,:\,k\in\mathbb{Z}\}$, only 0 is contained in the inside of the circle $\partial D(0,1/2)$. So, by the Cauchy Residues Theorem, we have

$$\int_{\partial D(0,1/2)} \frac{(e^z+1)\cos z}{\sin(\pi z)}\,\mathrm{d}z = 2\pi i W(\partial D(0,1/2),0)\,\mathrm{Res}(f,0).$$

Note that $W(\partial D(0,1/2),0)=1$ because $\partial D(0,1/2)$ travels precisely one time and in the positive orientation around 0. Therefore,

$$\int_{\partial D(0,1/2)} \frac{(e^z+1)\cos z}{\sin(\pi z)}\,\mathrm{d}z = 2\pi i \operatorname{Res}(f,0).$$

Let's calculate the residue. We write

$$f(z)=\frac{g(z)}{h(z)};\quad g(z)=(e^z+1)\cos z,\quad h(z)=\sin(\pi z),\quad h'(z)=\pi\cos(\pi z).$$

When $z=0$, we have that

$$g(0)=2,\quad h(0)=0,\quad h'(0)=\pi\cos(0)=\pi.$$

By the Proposition, we have that

$$\operatorname{Res}(f,0)=\frac{g(0)}{h'(0)}=\frac{2}{\pi}.$$

We can conclude that

$$\int_{\partial D(0,1/2)} \frac{(e^z+1)\cos z}{\sin(\pi z)}\,\mathrm{d}z = 2\pi i\frac{2}{\pi}=4i.$$

(b) The following theorem takes care of integrals in $[0,2\pi]$ of bounded rational-trigonometric functions:

Theorem (Evaluation of Trigonometric Integrals)**.** *Let $R(u,v)$ be a rational function of two variables such that the function $[0,2\pi]\ni\theta\mapsto R(\cos\theta,\sin\theta)$ is bounded in $[0,2\pi]$. Consider the function*

$$f(z):=\frac{1}{iz}R\left(\frac{1}{2}\left(z+\frac{1}{z}\right),\frac{1}{2i}\left(z-\frac{1}{z}\right)\right),$$

and denote $\operatorname{Poles}(f):=\{z\in\mathbb{C} : f \text{ has a pole at } z\}$ *and $\mathbb{D}=D(0,1)$ the open unit disk. Then,*

$$\int_0^{2\pi} R(\cos\theta,\sin\theta)\,\mathrm{d}\theta = 2\pi i\sum_{z\in\mathbb{D}\cap\operatorname{Poles}(f)}\operatorname{Res}(f,z).$$

In our exercise, we have that

$$R(u,v)=\frac{1}{13+12u}.$$

And the function f is

$$f(z)=\frac{1}{iz}\cdot\frac{1}{13+12\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)}=\frac{1}{iz}\cdot\frac{1}{13+6\left(z+\frac{1}{z}\right)}=\frac{1}{iz}\cdot\frac{1}{\frac{6z^2+13z+6}{z}}=\frac{1}{i}\cdot\frac{1}{6z^2+13z+6}.$$

By the theorem,

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{13+12\cos\theta}=2\pi i\sum_{z\in\mathbb{D}\cap\operatorname{Poles}(f)}\operatorname{Res}(f,z).$$

Now we need to indentify the singularities of f and select those that are contained in $\mathbb{D}$. The zeros of $6z^2+13z+6$ are

$$z=\frac{-13\pm\sqrt{169-144}}{12}=-\frac{3}{2},\,-\frac{2}{3}.$$

Thus we need to confirm that f has a pole in $-2/3$ and calculate the residue. The other singularity $-3/2$ is not in $\mathbb{D}$, so we ignore it. But observe that

$$f(z)=\frac{g(z)}{h(z)},\quad g(z)=1,\quad h(z)=i\left(6z^2+13z+6\right),\quad h'(z)=i\left(12z+13\right).$$

We have that $g(-2/3) = 1 \neq 0$, $h(-2/3) = 0$, $h'(-2/3) = 5i \neq 0$. By the proposition,

$$\operatorname{Res}(f, -2/3) = \frac{1}{5}.$$

We can conclude

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{13 + 12\cos\theta} = 2\pi i \sum_{z \in \mathbb{D} \cap \operatorname{Poles}(f)} \operatorname{Res}(f, z) = 2\pi i \operatorname{Res}(f, -2/3) = 2\pi i \frac{1}{5i} = \frac{2\pi}{5}.$$

(c) For this type of integral, we use:

Theorem. *Denote $\mathbb{H} := \{z \in \Omega : \operatorname{Im}(z) \geq 0\}$ the upper half-plane and let $\Omega \subset \mathbb{C}$ be open with $\mathbb{H} \subset \Omega$. Let f be a function with the following conditions*

- *f is holomorphic in Ω except at finitely many singularities $z_1, \ldots, z_N \in \Omega$.*
- *$z_k \notin \mathbb{R}$ (that is, $\operatorname{Im}(z_k) \neq 0$) for all $k = 1, \ldots, N$.*
- *There are constants $M, R_0 > 0$, $p > 0$ such that $|f(z)| \leq \frac{M}{|z|^p}$ for all $z \in \Omega$ with $|z| > R_0$.*

Then, for all $a > 0$, if $g(z) := f(z)e^{iaz}$, $z \in \Omega$, we have

$$\operatorname{pv} \int_{-\infty}^{+\infty} f(x) e^{iax} \,\mathrm{d}x = 2\pi i \sum_{\{k \,:\, \operatorname{Im}(z_k) > 0\}} \operatorname{Res}(g, z_k).$$

In our problem,

$$f(z) = \frac{z^2}{z^4 + 1}, \quad g(z) = e^{iaz} f(z).$$

The singularities of f are those $z \in \mathbb{C}$ such that $z^4 = -1$, that is

$$\{z_0 := e^{\pi i/4},\ z_1 := e^{3\pi i/4},\ z_2 := e^{5\pi i/4},\ z_3 := e^{7\pi i/4}\}.$$

So, f has finitely many singularities, none of them in the real line. Also, f is quotient of the polynomials z^2 and $1 + z^4$, where $\deg(1 + z^4) \geq 1 + \deg(z^2)$. Thue, the theorem applies for our problem, and so

$$\operatorname{pv} \int_{-\infty}^{+\infty} f(x) e^{iax} \,\mathrm{d}x = 2\pi i \sum_{\{k \,:\, \operatorname{Im}(z_k) > 0\}} \operatorname{Res}(g, z_k).$$

So, we only need to examine those singularities whose imaginary part is positive. Therefore, we only need to look at $z_0 = e^{\pi i/4}$ and $z_1 = e^{3\pi i/4}$. And we now calculate the residues of g (not of f!) at those points. Observe that

$$g(z) = \frac{\varphi(z)}{\psi(z)}, \quad \varphi(z) = z^2 e^{iaz}, \quad \psi(z) = z^4 + 1, \quad \psi'(z) = 4z^3.$$

For the first singularity z_0, we have $\varphi(z_0) \neq 0$, $\psi(z_0) = 0$ and $\psi'(z_0) \neq 0$. Our proposition for the calculus of residues tells us that g is a pole of order 1 at z_0, with

$$\operatorname{Res}(g, z_0) = \frac{\varphi(z_0)}{\psi'(z_0)} = \frac{z_0^2 e^{iaz_0}}{4z_0^3} = \frac{e^{iaz_0}}{4z_0}.$$

The exact same argument shows that

$$\operatorname{Res}(g, z_1) = \frac{e^{iaz_1}}{4z_1}.$$

$$\operatorname{pv} \int_{-\infty}^{+\infty} f(x) e^{iax} \,\mathrm{d}x = 2\pi i \sum_{\{k \,:\, \operatorname{Im}(z_k) > 0\}} \operatorname{Res}(g, z_k) = 2\pi i \left(\frac{e^{iaz_0}}{4z_0} + \frac{e^{iaz_1}}{4z_1} \right).$$

To simplify the above expression we can for example notice that $z_1 = -\overline{z_0}$ and the above becomes

$$\begin{aligned}
\frac{\pi i}{2}\left(\frac{e^{iaz_0}}{z_0} - \frac{e^{iaz_1}}{\overline{z_0}}\right) &= \frac{\pi i}{2}\left(\frac{e^{iaz_0}}{z_0} - \frac{e^{iaz_1}}{\overline{z_0}}\right) = \frac{\pi i}{2}\left(\frac{e^{\frac{ia}{\sqrt{2}}}e^{-\frac{a}{\sqrt{2}}}}{z_0} - \frac{e^{-\frac{ia}{\sqrt{2}}}e^{-\frac{a}{\sqrt{2}}}}{\overline{z_0}}\right) \\
&= \frac{\pi i e^{-\frac{a}{\sqrt{2}}}}{2|z_0|^2}\left(e^{\frac{ia}{\sqrt{2}}}\overline{z_0} - e^{-\frac{ia}{\sqrt{2}}}z_0\right) = \frac{\pi i e^{-\frac{a}{\sqrt{2}}}}{2} 2i \operatorname{Im}\left(e^{\frac{ia}{\sqrt{2}}}\overline{z_0}\right) \\
&= -\pi e^{-\frac{a}{\sqrt{2}}} \operatorname{Im}\left(e^{\frac{ia}{\sqrt{2}}}\overline{z_0}\right) = -\pi e^{-\frac{a}{\sqrt{2}}}\left(\tfrac{1}{\sqrt{2}}\sin\left(\tfrac{a}{\sqrt{2}}\right) - \tfrac{1}{\sqrt{2}}\cos\left(\tfrac{a}{\sqrt{2}}\right)\right) \\
&= \frac{\pi e^{-\frac{a}{\sqrt{2}}}}{\sqrt{2}}\left(\cos\left(\tfrac{a}{\sqrt{2}}\right) - \sin\left(\tfrac{a}{\sqrt{2}}\right)\right).
\end{aligned}$$

Thus,

$$\mathrm{pv}\int_{-\infty}^{+\infty} f(x)e^{iax}\,\mathrm{d}x = \frac{\pi e^{-\frac{a}{\sqrt{2}}}}{\sqrt{2}}\left(\cos\left(\tfrac{a}{\sqrt{2}}\right) - \sin\left(\tfrac{a}{\sqrt{2}}\right)\right).$$

Now, to get the original integral $\mathrm{pv}\int_{-\infty}^{+\infty}\frac{x^2\cos(ax)}{x^4+1}\,\mathrm{d}x$, we take the real part of the previous one, and get

$$\mathrm{pv}\int_{-\infty}^{+\infty}\frac{x^2\cos(ax)}{x^4+1}\,\mathrm{d}x = \frac{\pi e^{-\frac{a}{\sqrt{2}}}}{\sqrt{2}}\left(\cos\left(\tfrac{a}{\sqrt{2}}\right) - \sin\left(\tfrac{a}{\sqrt{2}}\right)\right).$$

□

Exercise 5 (1 point)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function with the property that

$$|f'(z)| \leq |z|, \quad \text{for all} \quad z \in \mathbb{C}.$$

Show that there exist numbers $w_0, w_2 \in \mathbb{C}$ with $|w_2| \leq 1/2$ so that $f(z) = w_0 + w_2 z^2$ for all $z \in \mathbb{C}$.

Suggestion: First use the Cauchy Estimates for the derivatives of f' to show that f' is a polynomial of degree 1*.*

Solution: Recall the Cauchy Estimates for the derivatives of holomorphic functions in $\mathbb{C}$:

Theorem. *Let $g : \mathbb{C} \to \mathbb{C}$ be holomorphic. Then, for every $n \in \mathbb{N}$, $r > 0$, and $z \in \mathbb{C}$: one has*

$$\left|g^{(n)}(z)\right| \leq \frac{n!}{r^n}\sup\{|g(w)| \,:\, |w - z| = r\}.$$

For our problem, set $g := f'$, and use the Cauchy Estimates for g, $n = 2$, and $z \in \mathbb{C}$, $r > 0$ arbitrary:

$$\left|g''(z)\right| \leq \frac{2}{r^2}\sup\{|g(w)| \,:\, |w - z| = r\} \leq \frac{2}{r^2}\sup\{|w| \,:\, |w - z| = r\} \leq \frac{2(r + |z|)}{r^2}.$$

If we fix $z \in \mathbb{C}$, since this is valid for all $r > 0$, we can let $r \to \infty$, and the above estimate shows that $g''(z) = 0$. Therefore, since $z \in \mathbb{C}$ is also arbitrary, we deduce that $g'' \equiv 0$ in $\mathbb{C}$. Therefore, $f''' \equiv 0$, which means that f is a polynomial of degree 2, that is, there are numbers $w_0, w_1, w_2 \in \mathbb{C}$ for which

$$f(z) = w_0 + w_1 z + w_2 z^2, \quad z \in \mathbb{C}.$$

It only remains to show that $w_1 = 0$ and $|w_2| \leq 1/2$. By the assumption,

$$|z| \geq |f'(z)| = |w_1 + 2w_2 z|, \quad \text{for all} \quad z \in \mathbb{C}.$$

So, letting $z = 0$, the above gives $w_1 = 0$. And letting $z = 1$, we then get $|w_2| \leq 1/2$, as desired. □

Exercise 6 (1 point)**.** Denote by $D(0,1)$ the open unit disk. Let $f : D(0,1) \to \mathbb{C}$ be a holomorphic function with the property that

$$f\left(\sin\left(\tfrac{1}{n}\right)\right) = \cos\left(\tfrac{2}{n}\right), \quad \text{for all} \quad n \in \mathbb{N}.$$

Give a formula for $f(z)$ for all $z \in D(0,1)$.

On the other hand, show that there is **no** holomorphic function $g : D(0,1) \to \mathbb{C}$ satisfying

$$g\left(\frac{1}{n}\right) = \frac{(-1)^n}{n^2}, \quad \text{for all} \quad n \in \mathbb{N},\, n \geq 2.$$

Suggestion: In both questions, use the Second Identity Principle for Holomorphic Functions.

Solution: Recall the Second Identity Principle For Holomorphic Maps:

Theorem. *Let $\Omega \subset \mathbb{C}$ be open and connected, and $f, g : \Omega \to \mathbb{C}$ two holomorphic functions such that there are $z_0 \in \Omega$ and a sequence $\{z_k\}_k \subset \Omega \setminus \{z_0\}$ such that $\lim_{k\to\infty} z_k = z_0$ and $f(z_k) = g(z_k)$ for all $k \in \mathbb{N}$. Then $f = g$ on Ω.*

In our exercise, we first recall the formula $\cos(2\theta) = 1 - 2\sin^2\theta$, which if you don't remember, you can use De Moivre's theorem to refresh it:

$$\cos(2\theta) + i\sin(2\theta) = e^{2i\theta} = \left(e^{i\theta}\right)^2 = (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta;$$

and so $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$. Therefore, for f we have that

$$f\left(\sin\left(\tfrac{1}{n}\right)\right) = \cos\left(\tfrac{2}{n}\right) = 1 - 2\sin^2\left(\tfrac{1}{n}\right) = 1 - 2\left(\sin\left(\tfrac{1}{n}\right)\right)^2.$$

This tells us that if $h(z) := 1 - 2z^2$, then $f = h$ at all points of the sequence $\{\sin\left(\frac{1}{n}\right)\}_{n\in\mathbb{N}}$. Note that $h : \mathbb{C} \to \mathbb{C}$ is holomorphic, and that $\{\sin\left(\frac{1}{n}\right)\}_{n\in\mathbb{N}}$ converges to 0, as $n \to \infty$, with $\sin\left(\frac{1}{n}\right) \neq 0$ for all $n \in \mathbb{N}$. By the Second Identity Principle, we get that

$$f(z) = h(z) = 1 - 2z^2, \quad \text{for all} \quad z \in \mathbb{C}.$$

For the second part, assume, for the sake of contradiction, that there is such a holomorphic function $g : D(0,1) \to \mathbb{C}$ with $g\left(\frac{1}{n}\right) = \frac{(-1)^n}{n^2}$ for all $n \in \mathbb{N}$. If we consider the subsequence $\{\frac{1}{2n}\}_{n\in\mathbb{N}}$, we see that

$$g\left(\frac{1}{2n}\right) = \frac{(-1)^{2n}}{(2n)^2} = \frac{1}{(2n)^2}, \quad \text{for all} \quad n \in \mathbb{N}.$$

That is, g coincides with the function $\varphi(z) = z^2$ at each term of the sequence $\{\frac{1}{2n}\}_n$. Since $\{\frac{1}{2n}\}_{n\in\mathbb{N}}$ converges to 0, and satisfies $\frac{1}{2n} \neq 0$ for all $n \in \mathbb{N}$, the Second Identity Principle implies that $g(z) = z^2$ for all $z \in \mathbb{C}$. But then, considering for example $n = 3$, by our assumption we have that

$$g\left(\frac{1}{3}\right) = \frac{-1}{9},$$

contradicting that $g(z) = z^2$ for all points $z \in \mathbb{C}$. □

Exercise 7 (1'5 points)**.** Let $f : [0,\pi] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \frac{\pi}{2}x & \text{if } 0 \leq x \leq \pi/2, \\ (x-\pi)^2 & \text{if } \pi/2 \leq x \leq \pi. \end{cases}$$

(a) Calculate the Sine-Fourier Coefficients of f in $[0,\pi]$, that is,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin(nt)\,\mathrm{d}t, \quad n \in \mathbb{N}.$$

Then write down the Sine-Fourier Series $S(f)(x)$ of f for all $x \in [0,\pi]$.

(b) Show that f is Lipschitz and then conclude that the Fourier series $S(f)(x)$ of f converges to $f(x)$ for all $x \in [0,\pi]$.

(c) If f is the function above, consider the Heat Equation in $[0,\pi]$ with boundary conditions:

$$(\mathrm{P}) \equiv \begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,t) = \dfrac{\partial u}{\partial t}(x,t); & \text{if} \quad (x,t) \in (0,\pi)\times(0,+\infty) \\ u(0,t) = u(\pi,t) = 0 & \text{if} \quad t \in [0,+\infty) \\ u(x,0) = f(x) & \text{if} \quad x \in [0,\pi]. \end{cases}$$

Write down a solution $u(x,t)$ of (P) as a series of functions.

Clarification: *You do NOT need to justify why this series $u(x,t)$ is a solution of* (P).

Solution:

(a) In the computation of b_n, we split the integral into two parts, we begin with the integral over $[0,\pi/2]$.

$$\int_0^{\pi/2} \frac{\pi}{2} t\sin(nt)\,\mathrm{d}t = \left[-\frac{\pi}{2}t\frac{\cos(nt)}{n}\right]_{t=0}^{t=\pi/2} - \int_0^{\pi/2}\left(-\frac{\pi}{2}\frac{\cos(nt)}{n}\right)\mathrm{d}t$$

$$= -\frac{\pi^2}{4}\frac{\cos(n\frac{\pi}{2})}{n} + \left[\frac{\pi}{2}\frac{\sin(nt)}{n^2}\right]_{t=0}^{t=\pi/2} = \frac{\pi}{2}\frac{\sin(n\frac{\pi}{2})}{n^2} - \frac{\pi^2}{4}\frac{\cos(n\frac{\pi}{2})}{n}.$$

Now we compute the integral we need in $[\pi/2,\pi]$:

$$\int_{\pi/2}^{\pi}(t-\pi)^2\sin(nt)\,\mathrm{d}t = \left[-(t-\pi)^2\frac{\cos(nt)}{n}\right]_{t=\pi/2}^{t=\pi} - \int_{\pi/2}^{\pi}2(t-\pi)\left(-\frac{\cos(nt)}{n}\right)\mathrm{d}t$$

$$= \frac{\pi^2}{4}\frac{\cos(n\frac{\pi}{2})}{n} + \int_{\pi/2}^{\pi}2(t-\pi)\frac{\cos(nt)}{n}\,\mathrm{d}t = \frac{\pi^2}{4}\frac{\cos(n\frac{\pi}{2})}{n} + \left[2(t-\pi)\frac{\sin(nt)}{n^2}\right]_{t=\pi/2}^{t=\pi} - 2\int_{\pi/2}^{\pi}\frac{\sin(nt)}{n^2}\,\mathrm{d}t$$

$$= \frac{\pi^2}{4}\frac{\cos(n\frac{\pi}{2})}{n} + \pi\frac{\sin(n\frac{\pi}{2})}{n^2} + \left[2\frac{\cos(nt)}{n^3}\right]_{t=\pi/2}^{t=\pi} = \frac{\pi^2}{4}\frac{\cos(n\frac{\pi}{2})}{n} + \pi\frac{\sin(n\frac{\pi}{2})}{n^2} + 2\frac{\cos(n\pi)}{n^3} - 2\frac{\cos(n\frac{\pi}{2})}{n^3}$$

Therefore,

$$b_n = \frac{2}{\pi}f(t)\sin(nt)\,\mathrm{d}t = \frac{2}{\pi}\left(\int_0^{\pi/2}\frac{\pi}{2}t\sin(nt)\,\mathrm{d}t + \int_{\pi/2}^{\pi}(t-\pi)^2\sin(nt)\,\mathrm{d}t\right)$$

$$= \frac{2}{\pi}\left(\frac{\pi}{2}\frac{\sin(n\frac{\pi}{2})}{n^2} + \pi\frac{\sin(n\frac{\pi}{2})}{n^2} + 2\frac{\cos(n\pi)}{n^3} - 2\frac{\cos(n\frac{\pi}{2})}{n^3}\right)$$

$$= \frac{2}{\pi}\left(\frac{3\pi}{2}\frac{\sin(n\frac{\pi}{2})}{n^2} + 2\frac{\cos(n\pi)-\cos(n\frac{\pi}{2})}{n^3}\right).$$

The Sine Fourier Series of f in $[0,\pi]$ is

$$S(f)(x) = \sum_{n=1} b_n\sin(nx) = \frac{2}{\pi}\sum_{n=1} b_n\sin(nx) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{3\pi}{2}\frac{\sin(n\frac{\pi}{2})}{n^2} + 2\frac{\cos(n\pi)-\cos(n\frac{\pi}{2})}{n^3}\right)\sin(nx).$$

(b) We need to find $C > 0$ so that

$$|f(x) - f(y)| \leq C|x - y|, \quad x, y \in [0, \pi].$$

When a **continuous** function in an interval $[a, b]$ is defined in two (or finitely-many) pieces, and the function is Lipschitz on each piece, then the function is Lipschitz in $[a, b]$. Thus, it suffices to check that the functions $[0, \pi/2] \ni x \mapsto \frac{\pi}{2}x$, $[\pi/2, \pi] \ni x \mapsto (x - \pi)^2$ are Lipschitz. The first one:

$$\left|\frac{\pi}{2}x - \frac{\pi}{2}y\right| = \frac{\pi}{2}|x - y|, \quad x, y \in [0, \pi/2].$$

For the latter,

$$\left|(x - \pi)^2 - (y - \pi)^2\right| = |(x - \pi) + (y - \pi)|\,|(x - \pi) - (y - \pi)| \leq |x + y - 2\pi|\,|x - y| \leq 4\pi|x - y|,$$

for all $x, y \in [\pi/2, \pi]$. Thus f is Lipschitz in $[0, \pi]$.

And we know that then $S(f)(x) = f(x)$ for all $x \in [0, \pi]$; where

$$S(f)(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

(c) A solution $u(x, t)$ of the equation in series form is

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx) = \sum_{n=1}^{\infty} \frac{2}{\pi} \left(\frac{3\pi}{2} \frac{\sin(n\frac{\pi}{2})}{n^2} + 2\frac{\cos(n\pi) - \cos(n\frac{\pi}{2})}{n^3} \right) e^{-n^2 t} \sin(nx),$$

for all $(x, t) \in [0, \pi] \times [0, +\infty)$. $\square$

Some Relevant Formulas

- De Moivre's: $e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$.
- Complex Exponential: $e^{x+iy} = e^x \cdot e^{iy}$.
- Complex Trigonometric functions:

$$\cos w := \frac{e^{iw} + e^{-iw}}{2}, \quad \sin w := \frac{e^{iw} - e^{-iw}}{2i}, \quad w \in \mathbb{C}.$$

- The Cauchy-Riemann Equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

- The Laplacian:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

- The Cauchy Integral Formula (under the right assumptions on f, γ, z):

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}} \, \mathrm{d}w, \quad n \in \mathbb{N} \cup \{0\}.$$

- Fourier Exponential Coefficients and Series, for a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ in $\mathbb{R}$.

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} \, \mathrm{d}t,$$

$$S(f)(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{inx}.$$

- Fourier Cosine and Sine Coefficients, for a function $f : [0, \pi] \to \mathbb{R}$:

$$a_0 := \frac{1}{\pi} \int_0^\pi f(t) \, \mathrm{d}t, \quad a_n := \frac{2}{\pi} \int_0^\pi f(t) \cos(nt) \, \mathrm{d}t, \quad b_n := \frac{2}{\pi} \int_0^\pi f(t) \sin(nt) \, \mathrm{d}t, \quad n \in \mathbb{N}.$$

- For $f : [0, \pi] \to \mathbb{R}$, the Cosine Fourier Series is

$$S(f)(x) = a_0 + \sum_{n=1}^\infty a_n \cos(nx).$$

- For $f : [0, \pi] \to \mathbb{R}$, the Sine Fourier Series is

$$S(f)(x) = \sum_{n=1}^\infty b_n \sin(nx).$$

- The Indefinite Integrals:

$$\int x^m \cos(bx) \, \mathrm{d}x = \frac{x^m \sin(bx)}{b} - \frac{m}{b} \int x^{m-1} \sin(bx) \, \mathrm{d}x$$

$$\int x^m \sin(bx) \, \mathrm{d}x = -\frac{x^m \cos(bx)}{b} + \frac{m}{b} \int x^{m-1} \cos(bx) \, \mathrm{d}x.$$