2^{nd} EXAM DRILL-MA2106. 19 NOVEMBER 2024

Exercise 1 (1 point)**.** For the function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = \sin\left(z^2\right), \quad z \in \mathbb{C},$$

find $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$, that is, the real and imaginary parts of f.

Exercise 2 (1'5 points)**.** For the function $u : \mathbb{R}^2 \to \mathbb{R}$ given by

$$u(x, y) = 3x^2y + 3x^2 - y^3 - 3y^2 - 3y + e^x \cos y, \quad (x, y) \in \mathbb{R}^2,$$

prove that u is harmonic in $\mathbb{R}^2$ by veryfing the laplace equation $\Delta u(x, y) = 0$, for all $(x, y) \in \mathbb{R}^2$. Then, find the harmonic conjugate v of u that satisfies $v(0, 0) = 1$.

Exercise 3 (1 point)**.** Use the Cauchy Integral Formula to evaluate the complex path-integral:

$$\int_{\partial D(0,1)} \frac{\sin(z^2)}{\left(z + \frac{3}{2}\right)^2 \left(z - \frac{2}{3}\right)} \,\mathrm{d}z;$$

where $\partial D(0, 1)$ is traveled once and counterclockwise.

Exercise 4 (3 points)**.** Use the Cauchy Residues Theorem to evaluate:

(a) The complex path-integral:

$$\int_{\partial D(0,1/2)} \frac{(e^z + 1)\cos z}{\sin(\pi z)} \,\mathrm{d}z;$$

where $\partial D(0, 1/2)$ is the circle centered at 0 and with radius 1/2, traveled once and counterclockwise.

(b) The real integral:

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{13 + 12\cos\theta}.$$

(c) The principal value of the real integral:

$$\mathrm{pv} \int_{-\infty}^{+\infty} \frac{x^2 \cos(ax)}{x^4 + 1} \,\mathrm{d}x,$$

for a $a \in \mathbb{R}$, $a > 0$.

Exercise 5 (1 point)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function with the property that

$$|f'(z)| \leq |z|, \quad \text{for all} \quad z \in \mathbb{C}.$$

Show that there exist numbers $w_0, w_2 \in \mathbb{C}$ with $|w_2| \leq 1/2$ so that $f(z) = w_0 + w_2 z^2$ for all $z \in \mathbb{C}$.

Suggestion: First use the Cauchy Estimates for the derivatives of f' to show that f' is a polynomial of degree 1*.*

Exercise 6 (1 point)**.** Denote by $D(0,1)$ the open unit disk. Let $f : D(0,1) \to \mathbb{C}$ be a holomorphic function with the property that

$$f\left(\sin\left(\tfrac{1}{n}\right)\right) = \cos\left(\tfrac{2}{n}\right), \quad \text{for all} \quad n \in \mathbb{N}.$$

Give a formula for $f(z)$ for all $z \in D(0,1)$.

On the other hand, show that there is **no** holomorphic function $g : D(0,1) \to \mathbb{C}$ satisfying

$$g\left(\frac{1}{n}\right) = \frac{(-1)^n}{n^2}, \quad \text{for all} \quad n \in \mathbb{N},\, n \geq 2.$$

Suggestion: In both questions, use the Second Identity Principle for Holomorphic Functions.

Exercise 7 (1'5 points)**.** Let $f : [0,\pi] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \frac{\pi}{2}x & \text{if } 0 \leq x \leq \pi/2, \\ (x-\pi)^2 & \text{if } \pi/2 \leq x \leq \pi. \end{cases}$$

(a) Calculate the Sine-Fourier Coefficients of f in $[0,\pi]$, that is,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin(nt)\,\mathrm{d}t, \quad n \in \mathbb{N}.$$

Then write down the Sine-Fourier Series $S(f)(x)$ of f for all $x \in [0,\pi]$.

(b) Show that f is Lipschitz and then conclude that the Fourier series $S(f)(x)$ of f converges to $f(x)$ for all $x \in [0,\pi]$.

(c) If f is the function above, consider the Heat Equation in $[0,\pi]$ with boundary conditions:

$$(\mathrm{P}) \equiv \begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,t) = \dfrac{\partial u}{\partial t}(x,t); & \text{if} \quad (x,t) \in (0,\pi) \times (0,+\infty) \\ u(0,t) = u(\pi,t) = 0 & \text{if} \quad t \in [0,+\infty) \\ u(x,0) = f(x) & \text{if} \quad x \in [0,\pi]. \end{cases}$$

Write down a solution $u(x,t)$ of (P) as a series of functions.

Clarification: *You do NOT need to justify why this series $u(x,t)$ is a solution of* (P).

Some Relevant Formulas

- De Moivre's: $e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$.
- Complex Exponential: $e^{x+iy} = e^x \cdot e^{iy}$.
- Complex Trigonometric functions:

$$\cos w := \frac{e^{iw} + e^{-iw}}{2}, \quad \sin w := \frac{e^{iw} - e^{-iw}}{2i}, \quad w \in \mathbb{C}.$$

- The Cauchy-Riemann Equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

- The Laplacian:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

- The Cauchy Integral Formula (under the right assumptions on f, γ, z):

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}} \, \mathrm{d}w, \quad n \in \mathbb{N} \cup \{0\}.$$

- Fourier Exponential Coefficients and Series, for a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ in $\mathbb{R}$.

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} \, \mathrm{d}t,$$

$$S(f)(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{inx}.$$

- Fourier Cosine and Sine Coefficients, for a function $f : [0,\pi] \to \mathbb{R}$:

$$a_0 := \frac{1}{\pi} \int_0^\pi f(t) \, \mathrm{d}t, \quad a_n := \frac{2}{\pi} \int_0^\pi f(t) \cos(nt) \, \mathrm{d}t, \quad b_n := \frac{2}{\pi} \int_0^\pi f(t) \sin(nt) \, \mathrm{d}t, \quad n \in \mathbb{N}.$$

- For $f : [0,\pi] \to \mathbb{R}$, the Cosine Fourier Series is

$$S(f)(x) = a_0 + \sum_{n=1}^\infty a_n \cos(nx).$$

- For $f : [0,\pi] \to \mathbb{R}$, the Sine Fourier Series is

$$S(f)(x) = \sum_{n=1}^\infty b_n \sin(nx).$$

- The Indefinite Integrals:

$$\int x^m \cos(bx) \, \mathrm{d}x = \frac{x^m \sin(bx)}{b} - \frac{m}{b} \int x^{m-1} \sin(bx) \, \mathrm{d}x$$

$$\int x^m \sin(bx) \, \mathrm{d}x = -\frac{x^m \cos(bx)}{b} + \frac{m}{b} \int x^{m-1} \cos(bx) \, \mathrm{d}x.$$