# EXAM DRILL-MA2106. 11 NOVEMBER 2024

**Exercise 1** (1 point)**.** For the function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = e^{z^3}, \quad z \in \mathbb{C},$$

find $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$, that is, the real and imaginary parts of $f$.

**Exercise 2** (1 point)**.** For the function $u : \mathbb{R}^2 \to \mathbb{R}$ given by

$$u(x, y) = y^3 - 3x^2y, \quad (x, y) \in \mathbb{R}^2,$$

prove that $u$ is harmonic in $\mathbb{R}^2$ and find the harmonic conjugate $v$ of $u$ that satisfies $v(0, 0) = 1$.

**Exercise 3** (1 point)**.** Consider the series of functions

$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{n^2}, \quad z \in \mathbb{C}.$$

Prove that:

(a) The series converges uniformly in $z \in \mathbb{R}$.

(b) For each $z \in \mathbb{C} \setminus \mathbb{R}$, the numerical series $\sum_{n=1}^{\infty} \frac{\sin(nz)}{n^2}$ diverges.

*Suggestion: In part (b), it might be helpful to study the modulus of the general term distinguishing the cases* $\operatorname{Im}(z) > 0$ *and* $\operatorname{Im}(z) < 0$.

**Exercise 4** (1 point)**.** Use the Cauchy Integral Formula to evaluate the complex path-integral:

$$\int_{\partial D(0,3)} \frac{z^2 + 7z + 6}{(z + 2)(z + 4)} \, \mathrm{d}z;$$

where $\partial D(0, 3)$ is traveled once and counterclockwise.

**Exercise 5** (2 points)**.** Use the Cauchy Residues Theorem to evaluate:

(a) The complex path-integral:

$$\int_{\gamma} \frac{\cos z}{z^2 - 4} \, \mathrm{d}z;$$

where $\gamma$ represents the ellipse $\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} = 1\}$, traveled once and counterclockwise.

(b) The principal value of the real integral:

$$\mathrm{pv} \int_{-\infty}^{+\infty} \frac{\sin x}{x^2 - 2x + 2} \, \mathrm{d}x.$$

**Exercise 6** (2 points)**.** Let $\mathbb{D} = D(0,1)$ be the unit disk, and let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic.

(a) Prove that the function $g : \mathbb{D} \to \mathbb{C}$ given by

$$g(z) = \overline{f(\overline{z})}, \quad \text{for all} \quad z \in \mathbb{D},$$

is holomorphic in $\mathbb{D}$ as well.

(b) Assume further that $f\left(\frac{1}{n+1}\right) \in \mathbb{R}$ for all $n \in \mathbb{N}$. Prove that then

$$f(z) = g(z), \quad \text{for all} \quad z \in \mathbb{D},$$

where $g$ is the function from (a).

*Suggestion: In part (b), use appropriately the Identity Principle II for Holomorphic Functions.*

**Exercise 7** (2 points)**.** Let $f : [0, \pi] \to \mathbb{R}$ be the function defined by

$$f(x) = \frac{\pi}{2} - \left| x - \frac{\pi}{2} \right|, \quad \text{for all} \quad x \in [0, \pi],$$

which we may consider extended as an odd function $f : [-\pi, \pi] \to \mathbb{R}$ in $[-\pi, \pi]$.

(a) Calculate the Fourier Coefficients of $f$ :

$$b_n := \frac{2}{\pi} \int_0^{\pi} f(t) \sin(nt)\, \mathrm{d}t, \quad n \in \mathbb{N}.$$

Write the Fourier Series (of Sines) $S(f)(x)$ of $f$ for all $x \in [0, \pi]$.

(b) Prove that $f$ is Lipschitz and then deduce that the Fourier series $S(f)(x)$ of $f$ converges to $f(x)$ for all $x \in [0, \pi]$.

(c) If $f$ is the function above, consider the Heat Equation in $[0, \pi]$ with boundary conditions:

$$(\mathrm{P}) \equiv \begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,t) = \dfrac{\partial u}{\partial t}(x,t); & \text{if} \quad (x,t) \in (0,\pi) \times (0,+\infty) \\ u(0,t) = u(\pi,t) = 0 & \text{if} \quad t \in [0,+\infty) \\ u(x,0) = f(x) & \text{if} \quad x \in [0,\pi]. \end{cases}$$

Based on the findings from (a), write down a solution $u(x,t)$ of (P) as a series of functions.

## Some Relevant Formulas

- De Moivre's: $e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$.
- Complex Trigonometric functions:

$$\cos w := \frac{e^{iw} + e^{-iw}}{2}, \quad \sin w := \frac{e^{iw} - e^{-iw}}{2i}, \quad w \in \mathbb{C}.$$

- The Cauchy-Riemann Equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

- The Cauchy Integral Formula (under the right assumptions on $f$, $\gamma$, $z$):

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}} \, \mathrm{d}w.$$

- Fourier Exponential Coefficients and Series, for a $2\pi$-periodic function $f : \mathbb{R} \to \mathbb{C}$ in $\mathbb{R}$.

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} \, \mathrm{d}t,$$

$$S(f)(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{inx}.$$

- Fourier Cosine-Sine Coefficients, for a function $f : [-\pi, \pi] \to \mathbb{R}$:

$$a_0 := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, \mathrm{d}t, \quad a_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) \, \mathrm{d}t, \quad b_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \, \mathrm{d}t, \quad n \in \mathbb{N},$$

$$S(f)(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos(nx) + b_n \sin(nx) \right).$$

- For $f : [-\pi, \pi] \to \mathbb{R}$ even,

$$S(f)(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx).$$

- For $f : [-\pi, \pi] \to \mathbb{R}$ odd,

$$S(f)(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$