FIRST EXAM DRILL SOLUTIONS-MA2106. 11 NOVEMBER 2024

Exercise 1 (1 point)**.** For the function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = e^{z^3}, \quad z \in \mathbb{C},$$

find $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$, that is, the real and imaginary parts of f.

Solution: We look for functions $u(x,y), v(x,y)$ in $\mathbb{R}^2$, for which

$$f(x+iy) = u(x,y) + iv(x,y), \quad (x,y) \in \mathbb{R}^2.$$

First, for $z = x+iy$, we write

$$\begin{aligned} z^3 &= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 \\ &= x^3 + 3x^2yi - 3xy^2 - iy^3 = x^3 - 3xy^2 + i\left(3x^2y - y^3\right). \end{aligned}$$

Now, the definition of the exponential $e^w = e^{\mathrm{Re}(w)} \cdot e^{i\,\mathrm{Im}(w)}$ gives

$$\begin{aligned} e^{z^3} &= e^{x^3-3xy^2} \cdot e^{i\left(3x^2y-y^3\right)} = e^{x^3-3xy^2}\left(\cos(3x^2y-y^3) + i\sin\left(3x^2y-y^3\right)\right) \\ &= e^{x^3-3xy^2}\cos(3x^2y-y^3) + ie^{x^3-3xy^2}\sin(3x^2y-y^3). \end{aligned}$$

We thus have that

$$\mathrm{Re}(f)(x,y) = u(x,y) = e^{x^3-3xy^2}\cos(3x^2y-y^3), \quad \mathrm{Im}(f)(x,y) = v(x,y) = e^{x^3-3xy^2}\sin(3x^2y-y^3).$$

□

Exercise 2 (1 point)**.** For the function $u : \mathbb{R}^2 \to \mathbb{R}$ given by

$$u(x,y) = y^3 - 3x^2y, \quad (x,y) \in \mathbb{R}^2,$$

prove that u is harmonic in $\mathbb{R}^2$ and find the harmonic conjugate v of u that satisfies $v(0,0) = 1$.

Solution: We prove the harmonicity via the Laplace Equation $\Delta u = 0$. Indeed,

$$\frac{\partial u}{\partial x}(x,y) = -6xy, \quad \frac{\partial^2 u}{\partial x^2}(x,y) = -6y, \quad \frac{\partial u}{\partial y}(x,y) = 3y^2 - 3x^2, \quad \frac{\partial^2 u}{\partial y^2}(x,y) = 6y,$$

and so

$$\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = -6y + 6y = 0.$$

Thus, u is harmonic in $\mathbb{R}^2$. Now, recall that a harmonic conjugate is a function $v : \mathbb{R}^2 \to \mathbb{R}$ so that $u + iv$ is holomorphic. We can see that the function $z \mapsto iz^3$ satisfies

$$i(x+iy)^3 = i\left(x^3 - 3xy^2 + i\left(3x^2y - y^3\right)\right) = i\left(x^3 - 3xy^2\right) + y^3 - 3x^2y,$$

and thus all the harmonic conjugates of u are of the form

$$v(x,y) = x^3 - 3xy^2 + K, \quad K \in \mathbb{C}.$$

If we are not able to find that holomorphic function, we solve the following system of (Cauchy-Riemann) Differential Equations:

$$\frac{\partial v}{\partial y}(x,y) = \frac{\partial u}{\partial x}(x,y) = -6xy$$
$$\frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y) = -3y^2 + 3x^2.$$

Integrating with respect to y in the first one, we see that

$$v(x,y) = \int (-6xy)\,\mathrm{d}y = -3xy^2 + \varphi(x),$$

for some differentiable $\varphi : \mathbb{R} \to \mathbb{R}$. Plugging this into the second equation leads to

$$-3y^2 + \varphi'(x) = -3y^2 + 3x^2 \implies \varphi'(x) = 3x^2, \quad \varphi(x) = x^3 + K; \quad K \in \mathbb{C}.$$

Now that we know the expression for $\varphi(x)$ we use it in the expression for v :

$$v(x,y) = -3xy^2 + x^3 + K.$$

Since we are interested in that v that satisfies $v(0,0) = 0$, we see that necessarily $K = 0$, whence

$$v(x,y) = -3xy^2 + x^3.$$

□

Exercise 3 (1 point)**.** Consider the series of functions

$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{n^2}, \quad z \in \mathbb{C}.$$

Prove that:

(a) The series converges uniformly in $z \in \mathbb{R}$.
(b) For each $z \in \mathbb{C} \setminus \mathbb{R}$, the numerical series $\sum_{n=1}^{\infty} \frac{\sin(nz)}{n^2}$ diverges.

Suggestion: In part (b), it might be helpful to study the modulus of the general term distinguishing the cases $\mathrm{Im}(z) > 0$ *and* $\mathrm{Im}(z) < 0$.

Solution:

(a) Recall the Weierstrass M-test:

Theorem. *Let $A \subset \mathbb{C}$, and a sequence of functions $f_n : A \to \mathbb{C}$ such that for every $n \in \mathbb{N}$ there exists $M_n > 0$ with $|f_n(z)| \le M_n$ for all $z \in A$, and so that $\sum_{n=1}^{\infty} M_n$ is finite. Then the series $\sum_{n=1}^{\infty} f_n$ converges absolutely–uniformly on A.*

Let's apply this to our exercise. Observe that $f_n(z) = \frac{\sin(nz)}{n^2}$ satisfies, **for all** $z \in \mathbb{R}$

$$|f_n(z)| = \left|\frac{\sin(nz)}{n^2}\right| \le \frac{1}{n^2} =: M_n,$$

where $\sum_{n=1}^{\infty} M_n < \infty$. By the M-test, the series converges uniformly in $z \in \mathbb{R}$.

(b) Let $z \in \mathbb{C} \setminus \mathbb{R}$, write $z = x + iy$ and assume first that $y > 0$. Notice that

$$\sin(nz) = \frac{1}{2i}\left(e^{in(x+iy)} - e^{-in(x+iy)}\right) = \frac{1}{2i}\left(e^{inx}e^{-ny} - e^{-inx}e^{ny}\right).$$

Thus, using that $y > 0$, we write

$$\left|\frac{\sin(nz)}{n^2}\right| = \frac{\left|e^{inx}e^{-ny} - e^{-inx}e^{ny}\right|}{2n^2} \geq \frac{|e^{-inx}e^{ny}| - |e^{inx}e^{-ny}|}{2n^2}$$
$$= \frac{|e^{-inx}||e^{ny}| - ||e^{inx}||e^{-ny}|}{2n^2} = \frac{e^{ny} - e^{-ny}}{2n^2}.$$

But notice the the limit as $n \to \infty$ of the last term goes to infinity, and does the series of complex numbers

$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{n^2}$$

diverges. Recall the limit of the general term of a convergent series is always 0.

Now, if $y < 0$, we observe that

$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{n^2} = \sum_{n=1}^{\infty} -\frac{\sin(n(-z))}{n^2},$$

and both series have the same character (one diverges if and only if the other diverges). By $\mathrm{Im}(-z) = -y > 0$, and by what we have proved for those complex numbers with positive imaginary part, we deduce that the series (both of them) diverges. $\square$

Exercise 4 (1 point)**.** Use the Cauchy Integral Formula to evaluate the complex path-integral:

$$\int_{\partial D(0,3)} \frac{z^2 + 7z + 6}{(z+2)(z+4)} \,\mathrm{d}z;$$

where $\partial D(0,3)$ is traveled once and counterclockwise.

Solution: Recall the Cauchy Integral Formula in a Convex Domain:

Theorem (Cauchy Integral Formula for Convex Sets)**.** *Let $\Omega \subset \mathbb{C}$ be a convex open set, $\gamma : [a,b] \to \Omega$ a closed piecewise C^1-path and $f : \Omega \to \mathbb{C}$ holomorphic. Then,*

$$W(\gamma, z) f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \,\mathrm{d}w, \quad \textit{for all} \quad z \in \Omega \setminus \gamma^*.$$

In our exercise, consider $\Omega = D(0, 3 + \frac{1}{2})$, and

$$f(z) = \frac{z^2 + 7z + 6}{z + 4}, \quad z \in \Omega.$$

This way, f is holomorphic in the convex domain Ω, (as $-4 \notin \Omega$) and if we let γ to be the circle of radius 3 and centered at 0, traveled once and counterclockwise, we have that the trace γ^* of γ is entirely contained in Ω and that $W(\gamma, -2) = 1$. The Cauchy Integral Formula says that then

$$\int_{\partial D(0,3)} \frac{z^2 + 7z + 6}{(z+2)(z+4)} \,\mathrm{d}z = \int_{\partial D(0,3)} \frac{f(z)}{z+2} \,\mathrm{d}z = 2\pi i W(\gamma, -2) f(-2) = 2\pi i f(-2) = 2\pi i \frac{-4}{2} = -4\pi i.‘$$

$\square$

Exercise 5 (2 points)**.** Use the Cauchy Residues Theorem to evaluate:

(a) The complex path-integral:

$$\int_\gamma \frac{\cos z}{z^2-4}\,\mathrm{d}z;$$

where γ represents the ellipse $\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{9}+\frac{y^2}{4}=1\}$, traveled once and counterclockwise.

(b) The principal value of the real integral:

$$\mathrm{pv}\int_{-\infty}^{+\infty} \frac{\sin x}{x^2-2x+2}\,\mathrm{d}x.$$

Solution: Recall the Cauchy Residues Theorem:

Theorem (Cauchy Residues Theorem). *Let $f:\Omega\setminus\{z_1,\dots,z_N\}\to\mathbb{C}$ be a holomorphic function in Ω except at N distinct points $\{z_1,\dots,z_N\}\subset\Omega$, at which f has isolated singularities. Let $\gamma:[a,b]\to\Omega\setminus\{z_1,\dots,z_N\}$ be a closed and piecewise C^1-path with $W(\gamma,z)=0$ for all $z\notin\Omega$. Then, we have*

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i\sum_{k=1}^{N}\mathrm{Res}(f,z_k)W(\gamma,z_k).$$

Also, the following proposition is instrumental:

Proposition. *Let $g,h:\Omega\to\mathbb{C}$ holomorphic functions and $z_0\in\Omega$. Assume that:*

- *$g(z_0)\neq 0$, and*
- *$h(z_0)=0$ and $h'(z_0)\neq 0$.*

Then $f:=\dfrac{g}{h}$ has a simple pole (pole of order 1) at z_0 and

$$\mathrm{Res}(f,z_0)=\frac{g(z_0)}{h'(z_0)}.$$

(a) The function $f(z)=\frac{\cos(z)}{z^2-4}$ has singularities at $z=2,-2$. *The inside* of the ellipse γ is the set

$$\{(x,y)\in\mathbb{R}^2 : \tfrac{x^2}{9}+\tfrac{y^2}{4}<1\}.$$

We need to see whether our singularities ± 2 are in that set. We have that

$$\frac{2^2}{9}+\frac{0^2}{4}=\frac{4}{9}<1,$$

and so 2 is there. For the same reason, -2 is there as well. By the Cauchy Residues Theorem, we have

$$\int_\gamma \frac{\cos z}{z^2-4}\,\mathrm{d}z = 2\pi i\left[W(\gamma,2)\,\mathrm{Res}(f,2)+W(\gamma,-2)\,\mathrm{Res}(f,-2)\right].$$

Note that $W(\gamma,2)=W(\gamma,-2)=1$ because γ travels once and in the positive orientation around 2 and -2. Therefore,

$$\int_\gamma \frac{\cos z}{z^2-4}\,\mathrm{d}z = 2\pi i\left[\mathrm{Res}(f,2)+\mathrm{Res}(f,-2)\right].$$

Let's calculate those residues. We write

$$f(z)=\frac{g(z)}{h(z)};\quad g(z)=\cos z,\quad h(z)=z^2-4,\quad h'(z)=2z.$$

When $z=\pm 2$, we have that

$$g(\pm 2)=\cos 2\neq 0,\quad h(\pm 2)=0,\quad h'(\pm 2)=\pm 4\neq 0.$$

By the Proposition, we have that

$$\operatorname{Res}(f, \pm 2) = \frac{g(\pm 2)}{h'(\pm 2)} = \frac{\cos 2}{\pm 4} = \pm \frac{\cos 2}{4}.$$

We can conclude that

$$\int_\gamma \frac{\cos z}{z^2-4}\, \mathrm{d}z = 2\pi i \left[\operatorname{Res}(f,2) + \operatorname{Res}(f,-2)\right] = 2\pi i \left(\frac{\cos 2}{4} - \frac{\cos 2}{4} \right) = 0.$$

(b) For this type of integral, we use:

Theorem. *Denote* $\mathbb{H} := \{z \in \Omega : \operatorname{Im}(z) \geq 0\}$ *the upper half-plane and let* $\Omega \subset \mathbb{C}$ *be open with* $\mathbb{H} \subset \Omega$. *Let* f *be a function with the following conditions*

- f *is holomorphic in* Ω *except at finitely many singularities* $z_1, \ldots, z_N \in \Omega$.
- $z_k \notin \mathbb{R}$ *(that is,* $\operatorname{Im}(z_k) \neq 0$*) for all* $k = 1, \ldots, N$.
- *There are constants* $M, R_0 > 0$, $p > 0$ *such that* $|f(z)| \leq \frac{M}{|z|^p}$ *for all* $z \in \Omega$ *with* $|z| > R_0$.

Then, for all $a > 0$, *if* $g(z) := f(z)e^{iaz}$, $z \in \Omega$, *we have*

$$\operatorname{pv} \int_{-\infty}^{+\infty} f(x) e^{iax}\, \mathrm{d}x = 2\pi i \sum_{\{k \,:\, \operatorname{Im}(z_k) > 0\}} \operatorname{Res}(g, z_k).$$

The third condition is satisfied, for example, when $f = P/Q$ where P, Q are polynomials with $\deg(Q) \geq 1 + \deg(P)$.

In our exercise, we first evaluate the integral

$$\operatorname{pv} \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2 - 2x + 2}\, \mathrm{d}x.$$

Towards the application of the previous theorem, write $g(z) = \frac{e^{iz}}{z^2-2z+2}$ and $f(z) = \frac{1}{z^2-2z+2}$. We see that the solutions to $z^2 - 2z + 2 = 0$ are $1+i$ and $1-i$, and so f has singularities at $1 \pm i$. But only $1+i$ is in the upper half plane. Also f is quotient of two polynomials, where the denominator Q has degree $1 + \deg(P)$; where $P \equiv 1$ is the numerator. By the Theorem

$$\operatorname{pv} \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2 - 2x + 2}\, \mathrm{d}x = 2\pi i \operatorname{Res}(g, 1+i).$$

To calculate those residues, we use the Proposition above. Write

$$g(z) = \frac{u(z)}{v(z)}, \quad u(z) = e^{iz}, \quad v(z) = z^2 - 2z + 2, \quad v'(z) = 2z - 2,$$

and note that

$$u(1+i) = e^{i(1+i)} = e^{i-1} \neq 0, \quad v(1+i) = 0, \quad v'(1+i) = 2(1+i) - 2 = 2i.$$

By the proposition

$$\operatorname{Res}(g, 1+i) = \frac{u(1+i)}{v'(1+i)} = \frac{e^{i-1}}{2i},$$

and hence

$$\operatorname{pv} \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2 - 2x + 2}\, \mathrm{d}x = 2\pi i \operatorname{Res}(g, 1+i) = 2\pi i \frac{e^{i-1}}{2i} = \frac{\pi}{e} e^i = \frac{\pi}{e} (\cos 1 + i \sin 1).$$

Therefore, the requested integral is the imaginary part of the previous one:

$$\operatorname{pv} \int_{-\infty}^{+\infty} \frac{\sin x}{x^2 - 2x + 2}\, \mathrm{d}x = \operatorname{Im}\left(\operatorname{pv} \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2 - 2x + 2}\, \mathrm{d}x \right) = \frac{\pi \sin 1}{e}.$$

$\square$

Exercise 6 (2 points)**.** Let $\mathbb{D} = D(0,1)$ be the unit disk, and let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic.

(a) Prove that the function $g : \mathbb{D} \to \mathbb{C}$ given by

$$g(z) = \overline{f(\overline{z})}, \quad \text{for all} \quad z \in \mathbb{D},$$

is holomorphic in $\mathbb{D}$ as well.

(b) Assume further that $f\left(\frac{1}{n+1}\right) \in \mathbb{R}$ for all $n \in \mathbb{N}$. Prove that then

$$f(z) = g(z), \quad \text{for all} \quad z \in \mathbb{D},$$

where g is the function from (a).

Suggestion: In part (b), use appropriately the Identity Principle II for Holomorphic Functions.

Solution:

(a) Define $g : \mathbb{D} \to \mathbb{C}$ by $g(z) = \overline{f(\overline{z})}$ for all $z \in \mathbb{D}$. If $z_0 \in \mathbb{D}$, the limit defining the derivative is

$$\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{\overline{f(\overline{z})} - \overline{f(\overline{z_0})}}{z - z_0} = \lim_{z \to z_0} \frac{\overline{f(\overline{z}) - f(\overline{z_0})}}{z - z_0} = \lim_{z \to z_0} \overline{\left(\frac{f(\overline{z}) - f(\overline{z_0})}{\overline{z} - \overline{z_0}} \right)}$$

$$= \overline{\left(\lim_{z \to z_0} \frac{f(\overline{z}) - f(\overline{z_0})}{\overline{z} - \overline{z_0}} \right)} = \overline{\left(\lim_{w \to \overline{z_0}} \frac{f(w) - f(\overline{z_0})}{w - \overline{z_0}} \right)} = \overline{f'(\overline{z_0})} \in \mathbb{C}.$$

Thus, the limit exists, and this means that g is complex differentiable at z_0. Because $z_0 \in \mathbb{D}$ is arbitrary, we conclude that g is holomorphic in $\mathbb{D}$.

(b) Recall the Second Identity Principle For Holomorphic Maps:

Theorem. *Let $\Omega \subset \mathbb{C}$ be open and connected, and $f, g : \Omega \to \mathbb{C}$ two holomorphic functions such that there are $z_0 \in \Omega$ and a sequence $\{z_k\}_k \subset \Omega \setminus \{z_0\}$ such that $\lim_{k \to \infty} z_k = z_0$ and $f(z_k) = g(z_k)$ for all $k \in \mathbb{N}$. Then $f = g$ on Ω.*

In our exercise, by the assumption we have $f\left(\frac{1}{n+1}\right) \in \mathbb{R}$, and therefore, looking at the definition of g in (a):

$$g\left(\frac{1}{n+1}\right) = \overline{f\left(\overline{\frac{1}{n+1}}\right)} = \overline{f\left(\frac{1}{n+1}\right)} = f\left(\frac{1}{n+1}\right).$$

Since $\{\frac{1}{n+1}\}_n \to 0 \in \mathbb{D}$, and $\frac{1}{n+1} \neq 0$ for all $n \in \mathbb{N}$, the Second Identity Principle implies that $f(z) = g(z)$ for all $z \in \mathbb{D}$. $\square$

Exercise 7 (2 points)**.** Let $f : [0, \pi] \to \mathbb{R}$ be the function defined by

$$f(x) = \frac{\pi}{2} - \left| x - \frac{\pi}{2} \right|, \quad \text{for all} \quad x \in [0, \pi],$$

which we may consider extended as an odd function $f : [-\pi, \pi] \to \mathbb{R}$ in $[-\pi, \pi]$.

(a) Calculate the Fourier Coefficients of f :

$$b_n := \frac{2}{\pi} \int_0^\pi f(t) \sin(nt)\, \mathrm{d}t, \quad n \in \mathbb{N}.$$

Write the Fourier Series (of Sines) $S(f)(x)$ of f for all $x \in [0, \pi]$.

(b) Prove that f is Lipschitz and then deduce that the Fourier series $S(f)(x)$ of f converges to $f(x)$ for all $x \in [0, \pi]$.

(c) If f is the function above, consider the Heat Equation in $[0,\pi]$ with boundary conditions:

$$(\mathrm{P}) \equiv \begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,t) = \dfrac{\partial u}{\partial t}(x,t); & \text{if} \quad (x,t) \in (0,\pi) \times (0,+\infty) \\ u(0,t) = u(\pi,t) = 0 & \text{if} \quad t \in [0,+\infty) \\ u(x,0) = f(x) & \text{if} \quad x \in [0,\pi]. \end{cases}$$

Based on the findings from (a), write down a solution $u(x,t)$ of (P) as a series of functions.

<u>Solution</u>:

(a) The function f is

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le \pi/2 \\ \pi - x & \text{if } \pi/2 \le x \le \pi. \end{cases}$$

So, in the computation of b_n, we split the integral into two parts, we begin with the integral over $[0,\pi/2]$.

$$\int_0^{\pi/2} t\sin(nt)\,\mathrm{d}t = \left[-t\frac{\cos(nt)}{n}\right]_{t=0}^{t=\pi/2} - \int_0^{\pi/2} \left(-\frac{\cos(nt)}{n}\right)\mathrm{d}t$$

$$= -\frac{\pi}{2}\frac{\cos(n\frac{\pi}{2})}{n} + \left[\frac{\sin(nt)}{n^2}\right]_{t=0}^{t=\pi/2} = \frac{\sin(n\frac{\pi}{2})}{n^2} - \frac{\pi}{2}\frac{\cos(n\frac{\pi}{2})}{n}$$

Now we compute the integral we need in $[\pi/2,\pi]$:

$$\int_{\pi/2}^{\pi} (\pi - t)\sin(nt)\,\mathrm{d}t = \left[(\pi - t)\frac{\cos(nt)}{n}\right]_{t=\pi/2}^{t=\pi} - \int_{\pi/2}^{\pi} (-1)\left(-\frac{\cos(nt)}{n}\right)\mathrm{d}t$$

$$= \frac{\pi}{2}\frac{\cos(n\frac{\pi}{2})}{n} - \left[\frac{\sin(nt)}{n^2}\right]_{t=\pi/2}^{t=\pi} = \frac{\pi}{2}\frac{\cos(n\frac{\pi}{2})}{n} + \frac{\sin(n\frac{\pi}{2})}{n^2}.$$

And the requested coefficients are

$$b_n = \frac{2}{\pi}\left(\int_0^{\pi/2} t\sin(nt)\,\mathrm{d}t + \int_{\pi/2}^{\pi} (\pi - t)\sin(nt)\,\mathrm{d}t\right) = \frac{4}{\pi}\frac{\sin(n\frac{\pi}{2})}{n^2}.$$

The Fourier Series of Sines of f in $[0,\pi]$ is

$$\sum_{n=1}^{\infty} b_n \sin(nx) = \sum_{n=1}^{\infty} \frac{4}{\pi}\frac{\sin(n\frac{\pi}{2})}{n^2}\sin(nx), \quad x \in [0,\pi].$$

(b) The function f is Lipschitz in $[0,\pi]$, meaning that there is $C > 0$ constant with

$$|f(x) - f(y)| \le C|x - y| \quad \text{for all} \quad x,y \in [0,\pi].$$

Indeed, if $x,y \in [0,\pi]$:

$$|f(x) - f(y)| = \left|\frac{\pi}{2} - \left|x - \frac{\pi}{2}\right| - \left(\frac{\pi}{2} - \left|y - \frac{\pi}{2}\right|\right)\right| = \left|\left|y - \frac{\pi}{2}\right| - \left|x - \frac{\pi}{2}\right|\right| \le \left|y - \frac{\pi}{2} - \left(x - \frac{\pi}{2}\right)\right| = |x - y|.$$

We know that then the Fourier (Sine) Series

$$\sum_{n=1}^{\infty} \frac{4}{\pi}\frac{\sin(n\frac{\pi}{2})}{n^2}\sin(nx)$$

converges to $f(x)$ for all $x \in [0,\pi]$.

(c) Recall that a solution to this Heat Equation in $[0, \pi]$ is given by the Series of Functions:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx), \quad (x,t) \in [0,\pi] \times [0,+\infty);$$

where $\{b_n\}_{n \in \mathbb{N}}$ are the Sine Fourier Coefficients of the initial value f. In our case,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{\pi} \frac{\sin(n\frac{\pi}{2})}{n^2} e^{-n^2 t} \sin(nx).$$

$\square$

Some Relevant Formulas

- De Moivre's: $e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$.
- Complex Trigonometric functions:

$$\cos w := \frac{e^{iw} + e^{-iw}}{2}, \quad \sin w := \frac{e^{iw} - e^{-iw}}{2i}, \quad w \in \mathbb{C}.$$

- The Cauchy-Riemann Equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

- The Cauchy Integral Formula (under the right assumptions on f, γ, z):

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}} \, \mathrm{d}w.$$

- Fourier Exponential Coefficients and Series, for a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ in $\mathbb{R}$.

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} \, \mathrm{d}t,$$

$$S(f)(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{inx}.$$

- Fourier Cosine-Sine Coefficients, for a function $f : [-\pi, \pi] \to \mathbb{R}$:

$$a_0 := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, \mathrm{d}t, \quad a_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) \, \mathrm{d}t, \quad b_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \, \mathrm{d}t, \quad n \in \mathbb{N},$$

$$S(f)(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$

- For $f : [-\pi, \pi] \to \mathbb{R}$ even,

$$S(f)(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx).$$

- For $f : [-\pi, \pi] \to \mathbb{R}$ odd,

$$S(f)(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$