MA1301 FINAL EXAM – 2013

Problem 1: (a) Compute the greatest common divisor of 161 and 217.

(b) Determine all integers c for which

$$217x + 161y = c.$$

has integer solutions.

(c) Find all solutions to the following Diophantine equation:

$$217x + 161y = 14.$$

Problem 2: (a) Define for a number theoretic function the notion of multiplicativity.

(b) Define Euler's φ-function and the Möbius function μ.

(c) Compute $\varphi(60)$.

(d) Prove that μ is multiplicative.

Problem 3: (a) Suppose n is a positive integer. Let a be an integer relatively prime to n. Define the following notions: (i) the order of a modulo n; (ii) a primitive root of n.

(b) Determine the order of $1, 2, 3, 4, 5, 6$ modulo 7.

(c) Give arguments for the following facts: Among the integers $\{1, 2, 3, 4, 5, 6\}$ (i) there is 1 element of order 1, (ii) there is 1 element of order 2, (iii) there are 2 elements of order 3, (iv) 7 has 2 primitive roots.

Problem 4: (a) Suppose a is an integer relatively prime to an odd prime p. Define the following notions: (i) a is a quadratic residue modulo p; (ii) a is a quadratic residue modulo p; (iii) the Legendre symbol $\left(\frac{a}{p}\right)$.

(b) Show that if x_0 is a quadratic residue modulo p, then $p - x_0$ is also a quadratic residue.

(c) State the Quadratic Reciprocity Theorem.

(d) Compute $\left(\frac{-21}{37}\right)$ and explain each step of the computation.

Problem 5: Find all solutions of the system

$$\begin{aligned} x &\equiv 2 \mod 3 \\ x &\equiv 12 \mod 7 \\ x &\equiv 20 \mod 13 \end{aligned}$$