MA1202/6202

LØSNINGSFORSLAG TIL UTVALGTE OPPGAVER

SAMARBEIDSOPPGAVER S17

OPPGAVE S 17.1

a) Vi har

$$\|\bar{v}\|^2 = \langle \bar{v}, \bar{v} \rangle = \langle (1,i), (1,i) \rangle = 1 \cdot \bar{1} + i \cdot \bar{i}$$
$$= 1^2 - i^2 = 2,$$

så $\underline{\underline{\|\bar{v}\| = \sqrt{2}}}$.

b) Vektoren $\bar{u} = (i, 1) \in \mathbb{C}^2$ er ortgonal med $\bar{v}$

$$(\langle \bar{u}, \bar{v} \rangle = \langle (i,1), (1,i) \rangle = i \cdot \bar{1} + 1 \cdot \bar{i} = i - i = 0).$$

OPPGAVE S17.2

Her er $\bar{u} = (2,4,-1)$ og $\bar{v} = (1,1,1)$.

a) Vi bruker PROPOSISJON (ORTOGONAL DEKOMPONERING) fra FORELESNING E17: La

$$c = \frac{\langle \bar{u}, \bar{v} \rangle}{\|\bar{v}\|^2} = \frac{\langle (2,4,-1), (1,1,1) \rangle}{\langle (1,1,1), (1,1,1) \rangle} = \frac{2+4-1}{1+1+1} = \frac{5}{3}.$$

Da får vi den ønskede dekomponeringa ved å la

$$\bar{w} = \bar{u} - c\bar{v} = (2,4,-1) - \tfrac{5}{3}(1,1,1) = \tfrac{1}{3}(1,7,-8),$$

altså

$$\bar{u} = c\bar{v} + \bar{w} = \overbrace{\tfrac{5}{3}(1,1,1)}^{c\bar{v}} + \overbrace{\tfrac{1}{3}(1,7,-8)}^{\text{ortogonal med } \bar{v}}.$$

OPPGAVE S 17.2

Her er $\bar{u} = (2,4,-1)$ og $\bar{v} = (1,1,1)$.

b) Vi bruker PROPOSISJON (ORTOGONAL DEKOMPONERING) fra FORELESNING E17 igjen:

$$c = \frac{\langle \bar{u}, \bar{v} \rangle}{\|\bar{v}\|^2} = \frac{\langle (2,4,-1), (1,1,1) \rangle}{\langle 1, 1, 1), (1,1,1) \rangle} = \frac{1 \cdot 2 + 2 \cdot 4 + 3 \cdot (-1)}{1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1} = \frac{7}{6}.$$

Hvis vi lar

$$\bar{w} = \bar{u} - c\bar{v} = (2,4,-1) - \frac{7}{6}(1,1,1) = \frac{1}{6}(5, 17, -13),$$

så blir

$$\underline{\underline{\bar{u} = c\bar{v} + \bar{w} = \overbrace{\frac{7}{6}(1,1,1)}^{c\bar{v}} + \overbrace{\frac{1}{6}(5, 17, -13)}^{\text{ortogonal med } \bar{v}}.}}$$

OPPGAVE S 17.3

a) Dette er ikke et indreprodukt: For eksempel har vi en ikke-null vektor $\bar{v} = (0,1,0)$ med $\langle \bar{v}, \bar{v} \rangle = 0$, så aksiom ii) holder ikke.

b) Dette er ikke et indreprodukt: Se for eksempel på $\bar{u} = (1,1)$ og $\bar{v} = (-1,-1)$. Da er

$$\langle \bar{u} + \bar{v}, \bar{v} \rangle = \langle (1,1) + (-1,-1), (-1,-1) \rangle = \langle (0,0), (-1,-1) \rangle$$
$$= |0 \cdot (-1)| + |0 \cdot (-1)| = 0,$$

mens

$$\langle \bar{u}, \bar{v} \rangle + \langle \bar{v}, \bar{v} \rangle = \langle (1,1), (-1,-1) \rangle + \langle (1,1), (1,1) \rangle$$
$$= |1 \cdot (-1)| + |1 \cdot (-1)| + |1 \cdot 1| + |1 \cdot 1| = 4.$$

Altså er $\langle \bar{u}, \bar{v} \rangle + \langle \bar{v}, \bar{v} \rangle \neq \langle \bar{u} + \bar{v}, \bar{v} \rangle$, så aksiom iii) holder ikke.

OPPGAVE S 17.3

c) Her nøyer vi oss med å oppgi fasit:

Dette er et indreprodukt på $\mathbb{R}[x]$.

OPPGAVE S 17.4

a) Skal vise: $\quad |\langle \bar{u}, \bar{v} \rangle| \overset{(*)}{\leq} \|\bar{u}\| \, \|\bar{v}\|$

Hvis $\bar{v} = \bar{0}$, så holder $(*)$ opplagt $(0 \leq 0)$.

Så anta at $\bar{v} \neq \bar{0}$. Da har vi en ORTOGONAL DEKOMPONERING (FORELESNING E17), nemlig

$$\bar{u} \overset{(*)}{=} \frac{\langle \bar{u}, \bar{v} \rangle}{\|\bar{v}\|^2} \bar{v} + \bar{w}$$

med $\langle \bar{w}, \bar{v} \rangle \overset{(**)}{=} 0$.

OPPGAVE S 17.4

a) Nå følger det at

$$\|\bar{u}\|^2 \overset{(*)}{=} \left\| \frac{\langle \bar{u}, \bar{v} \rangle}{\|\bar{v}\|^2} \bar{v} + \bar{w} \right\|^2$$

PYTHAGORAS (FORELESNING V17)

$$\overset{(**)}{=} \left\| \frac{\langle \bar{u}, \bar{v} \rangle}{\|\bar{v}\|^2} \bar{v} \right\|^2 + \|\bar{w}\|^2$$

$$= \frac{|\langle \bar{u}, \bar{v} \rangle|^2}{\|\bar{v}\|^2} + \|\bar{w}\|^2 \geq \frac{|\langle \bar{u}, \bar{v} \rangle|^2}{\|\bar{v}\|^2}.$$

Dette betyr at $|\langle \bar{u}, \bar{v} \rangle| \overset{(*)}{\leq} \|\bar{u}\| \|\bar{v}\|$. $\square$

OPPGAVE S 17.4

b) Skal vise : $\|\bar{u}+\bar{v}\| \overset{(**)}{\leq} \|\bar{u}\| + \|\bar{v}\|$

Vi regner rett ut :

Konjugering i $\mathbb{C}$

AKSIOMER OG FUNDAMENTALE EGENSKAPER

AKSIOM v)

CAUCHY – SCHWARTZ (S17.4 a))

Realdelen til det komplekse tallet $\langle\bar{u},\bar{v}\rangle$

$$\begin{aligned}
\|\bar{u}+\bar{v}\|^2 &= \langle \bar{u}+\bar{v}, \bar{u}+\bar{v}\rangle \\
&= \langle\bar{u},\bar{u}\rangle + \langle\bar{v},\bar{v}\rangle + \langle\bar{u},\bar{v}\rangle + \langle\bar{v},\bar{u}\rangle \\
&= \langle\bar{u},\bar{u}\rangle + \langle\bar{v},\bar{v}\rangle + \langle\bar{u},\bar{v}\rangle + \overline{\langle\bar{u},\bar{v}\rangle} \\
&= \|\bar{u}\|^2 + \|\bar{v}\|^2 + 2\underbrace{\mathrm{Re}(\langle\bar{u},\bar{v}\rangle)} \\
&\leq \|\bar{u}\|^2 + \|\bar{v}\|^2 + 2|\langle\bar{u},\bar{v}\rangle| \\
&\leq \|\bar{u}\|^2 + \|\bar{v}\|^2 + 2\|\bar{u}\|\|\bar{v}\| \\
&= (\|\bar{u}\| + \|\bar{v}\|)^2 ,
\end{aligned}$$

og dette betyr at $\|\bar{u}+\bar{v}\| \overset{(**)}{\leq} \|\bar{u}\| + \|\bar{v}\|$. $\square$

OPPGAVE S 17.5

Skal vise: $16 \leq (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \quad \forall\, a,b,c,d>0.$

Så la $a,b,c,d>0$. Definér to vektorer $\bar{u},\bar{v}\in\mathbb{R}^4$ ved

$$\bar{u} = (\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}) \quad \text{og} \quad \bar{v} = \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{d}}\right).$$

I indreproduktrommet $\mathbb{R}^4$ med euklidske indreprodukt) blir

$$\|\bar{u}\|^2 \overset{(*)}{=} a+b+c+d, \quad \|\bar{v}\|^2 \overset{(**)}{=} \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \quad \text{og}$$

$$|\langle \bar{u},\bar{v}\rangle| = \sqrt{a}\frac{1}{\sqrt{a}} + \sqrt{b}\frac{1}{\sqrt{b}} + \sqrt{c}\frac{1}{\sqrt{c}} + \sqrt{d}\frac{1}{\sqrt{d}} \overset{(***)}{=} 4$$

Cauchy-Schwartz-ulikheten (S17.4 a)) gir nå

$$16 \overset{(***)}{=} |\langle \bar{u},\bar{v}\rangle|^2 \leq \|\bar{u}\|^2\|\bar{v}\|^2 \overset{(*)\&(**)}{=} (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right). \quad \square$$

OPPGAVE S 17.6

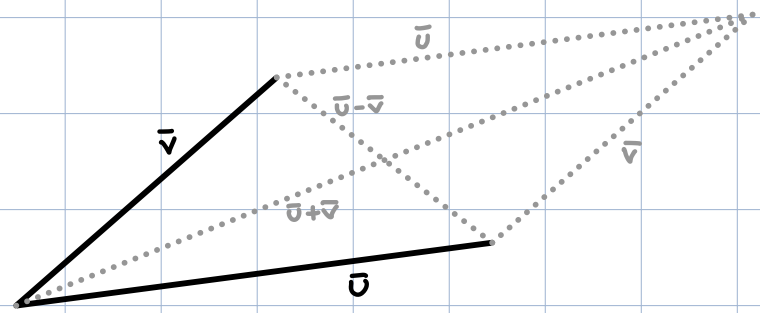
a) Skal vise: $\|\bar{u}+\bar{v}\|^2 + \|\bar{u}-\bar{v}\|^2 = 2\left(\|\bar{u}\|^2 + \|\bar{v}\|^2\right)$.

Dette er rettfrem:

$$
\begin{aligned}
\|\bar{u}+\bar{v}\|^2 + \|\bar{u}-\bar{v}\|^2 &= \langle \bar{u}+\bar{v}, \bar{u}+\bar{v}\rangle + \langle \bar{u}-\bar{v}, \bar{u}-\bar{v}\rangle \\
&= \langle \bar{u},\bar{u}\rangle + \langle \bar{u},\bar{v}\rangle + \langle \bar{v},\bar{u}\rangle + \langle \bar{v},\bar{v}\rangle \\
&\quad + \langle \bar{u},\bar{u}\rangle - \langle \bar{u},\bar{v}\rangle - \langle \bar{v},\bar{u}\rangle + \langle \bar{v},\bar{v}\rangle \\
&= 2\langle \bar{u},\bar{u}\rangle + 2\langle \bar{v},\bar{v}\rangle \\
&= 2\left(\|\bar{u}\|^2 + \|\bar{v}\|^2\right).
\end{aligned}
$$

$\square$

Etymologi:



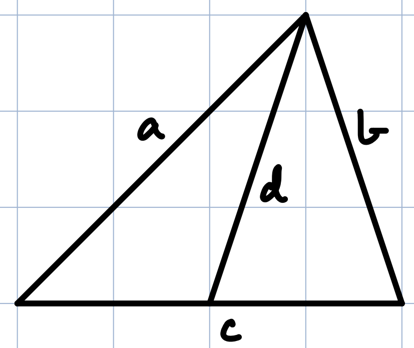
OPPGAVE S 17.6

b) Skal vise:
$$\left\| \bar{w} - \frac{1}{2}(\bar{u} + \bar{v}) \right\|^2 = \frac{\|\bar{w} - \bar{u}\|^2 + \|\bar{w} - \bar{v}\|^2}{2} - \frac{\|\bar{u} - \bar{v}\|^2}{4}.$$

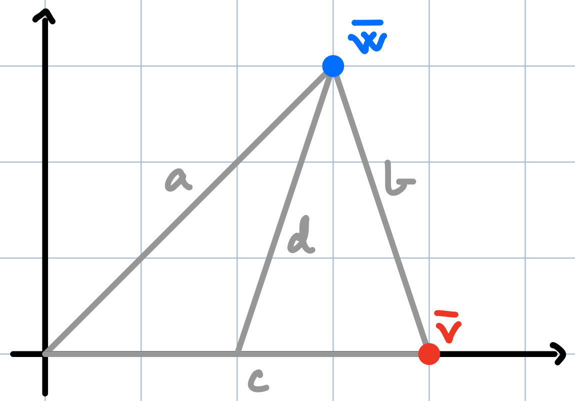
Vi regner i vei:
$$\left\| \bar{w} - \frac{1}{2}(\bar{u} + \bar{v}) \right\|^2 = \left\| \frac{\bar{w} - \bar{u}}{2} + \frac{\bar{w} - \bar{v}}{2} \right\|^2$$

PARALLELLOGRAM-LIGNINGA (S17.6a))
$$= 2\left\| \frac{\bar{w} - \bar{u}}{2} \right\|^2 + 2\left\| \frac{\bar{w} - \bar{v}}{2} \right\|^2 - \left\| \frac{\bar{w} - \bar{u}}{2} - \frac{\bar{w} - \bar{v}}{2} \right\|^2$$
$$= \frac{\|\bar{w} - \bar{u}\|^2 + \|\bar{w} - \bar{v}\|^2}{2} - \left\| \frac{-\bar{u} + v}{2} \right\|^2$$
$$= \frac{\|\bar{w} - \bar{u}\|^2 + \|\bar{w} - \bar{v}\|^2}{2} - \frac{\|\bar{u} - \bar{v}\|^2}{4} \qquad \square$$

OPPGAVE S17.7

Skal vise: $a^2 + b^2 = \frac{1}{2} c^2 + 2d^2$ for trekanten .

Se på problemet i $\mathbb{R}^2$ (med euklidsk indreprodukt):



$\|\bar{w}\| = a$; $\|\bar{v}\| = c$;

$\|\bar{w} - \bar{v}\| = b$

Da gir S17.6 b) (med $\bar{u} = \bar{0}$) at $a^2 + b^2 = \frac{1}{2} c^2 + 2d^2$. $\square$

OPPGAVE 5 17.8

a) Skal vise: $\langle \bar{u}, \bar{v} \rangle = \dfrac{\|\bar{u}+\bar{v}\|^2 - \|\bar{u}-\bar{v}\|^2}{4}$ når $F = \mathbb{R}$.

Dette regner vi lett ut:

$$\frac{\|\bar{u}+\bar{v}\|^2 - \|\bar{u}-\bar{v}\|^2}{4} = \frac{\langle \bar{u}+\bar{v}, \bar{u}+\bar{v} \rangle - \langle \bar{u}-\bar{v}, \bar{u}-\bar{v} \rangle}{4}$$

$$= \frac{\|\bar{u}\|^2 + 2\langle \bar{u}, \bar{v} \rangle + \|\bar{v}\|^2 - (\|\bar{u}\|^2 - 2\langle \bar{u}, \bar{v} \rangle + \|\bar{v}\|^2)}{4}$$

$$= \frac{2\langle \bar{u}, \bar{v} \rangle + 2\langle \bar{u}, \bar{v} \rangle}{4}$$

$$= \langle \bar{u}, \bar{v} \rangle.$$

$\square$

OPPGAVE S 17.8

b) Skal vise: $\langle \bar{u}, \bar{v} \rangle \overset{(*)}{=} \dfrac{\|\bar{u}+\bar{v}\|^2 - \|\bar{u}-\bar{v}\|^2 + \|\bar{u}+i\bar{v}\|^2 i - \|\bar{u}-i\bar{v}\|^2 i}{4}$

når $F = \mathbb{C}$.

Vi bruker aksiomene iii), iv) og v) samt FUNDAMENTALE EGENSKAPER fra FORELESNING V17 til å regne ut hvert av leddene i telleren

$$\|\bar{u}+\bar{v}\|^2 = \langle \bar{u}+\bar{v}, \bar{u}+\bar{v} \rangle = \|\bar{u}\|^2 + \langle \bar{u}, \bar{v} \rangle + \langle \bar{v}, \bar{u} \rangle + \|\bar{v}\|^2$$

$$-\|\bar{u}-\bar{v}\|^2 = -\langle \bar{u}-\bar{v}, \bar{u}-\bar{v} \rangle = -\|\bar{u}\|^2 + \langle \bar{u}, \bar{v} \rangle + \langle \bar{v}, \bar{u} \rangle - \|\bar{v}\|^2$$

OPPGAVE S 17.8

b)

$$
\begin{aligned}
\|\bar{u}+i\bar{v}\|^2 i &= i\langle \bar{u}+i\bar{v}, \bar{u}+i\bar{v}\rangle \\
&= i\left(\|\bar{u}\|^2 + \langle \bar{u}, i\bar{v}\rangle + \langle i\bar{v}, \bar{u}\rangle + \langle i\bar{v}, i\bar{v}\rangle\right) \\
&= i\left(\|\bar{u}\|^2 - i\langle \bar{u}, \bar{v}\rangle + i\langle \bar{v}, \bar{u}\rangle + i\|\bar{v}\|^2\right) \\
&= i\|\bar{u}\|^2 + \langle \bar{u}, \bar{v}\rangle - \langle \bar{v}, \bar{u}\rangle + i\|\bar{v}\|^2
\end{aligned}
$$

$$
\begin{aligned}
-\|\bar{u}-i\bar{v}\|^2 i &= -i\langle \bar{u}-i\bar{v}, \bar{u}-i\bar{v}\rangle \\
&= -i\left(\|\bar{u}\|^2 - \langle \bar{u}, iv\rangle - \langle i\bar{v}, \bar{u}\rangle + \|\bar{v}\|^2\right) \\
&= -i\left(\|\bar{u}\|^2 + i\langle \bar{u}, \bar{v}\rangle - i\langle \bar{v}, \bar{u}\rangle + \|\bar{v}\|^2\right) \\
&= -i\|\bar{u}\|^2 + \langle \bar{u}, \bar{v}\rangle - \langle \bar{v}, \bar{u}\rangle - i\|\bar{v}\|^2
\end{aligned}
$$

Nå er det kurant å se at

$$\|\bar{u}+\bar{v}\|^2 - \|\bar{u}-\bar{v}\|^2 + \|\bar{u}+i\bar{v}\|^2 i - \|\bar{u}-i\bar{v}\|^2 i = 4\langle \bar{u}, \bar{v}\rangle. \quad \square$$

OPPGAVE S 17.9

a) Det er rettfram å sjekke at $\langle -,- \rangle_A$ tilfredsstiller aksiomene i) – v) når man merker seg at

$$\langle \bar{x}, \bar{y} \rangle_A = \langle A\bar{x}, A\bar{y} \rangle_I = (A\bar{y})^* A\bar{x} = \bar{y}^* A^* A \bar{x} \quad \forall \bar{x}, \bar{y} \in F^n.$$

b) Det vekta euklidske indreproduktet på $\mathbb{R}^n$ med vektene $a_1, a_2, \ldots, a_n$ er det samme som indreproduktet $\langle -,- \rangle_A$ med

$$A = \begin{pmatrix} \sqrt{a_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{a_2} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{a_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sqrt{a_n} \end{pmatrix}$$

OPPGAVE S 17.9

c) $\langle -,- \rangle_f$ er et indreprodukt

Skal vise: $\iff$

f er injektiv

$\Rightarrow$: Anta at f ikke er injektiv. Da finnes en $\bar{v} \neq \bar{0}$ slik at $f(\bar{v}) = \bar{0}$ og da blir

$\langle -,- \rangle$ ER et indreprodukt på V

$$0 = \langle f(\bar{v}), f(\bar{v}) \rangle = \langle \bar{v}, \bar{v} \rangle_f \,,$$

så funksjonen $\langle -,- \rangle_f$ oppfyller ikke aksiom ii), så funksjonen $\langle -,- \rangle_f$ er ikke et indreprodukt.

$\Leftarrow$: Dette er helt rutinemessig. $\square$